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Equivalent Characterizations for Boundedness of Maximal Singular Integrals on $ax+b$–Groups

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Abstract Let (S,d,ρ) be the affine group $\mathbb{R}^n \ltimes \mathbb{R}^+$ endowed with the left-invariant Riemannian metric d and the right Haar measure ρ, which is of exponential growth at infinity. In this paper, for any linear operator T on (S,d,ρ) associated with a kernel K satisfying certain integral size condition and Hörmander's condition, the authors prove that the following four statements regarding the corresponding maximal singular integral T^* are equivalent: T^* is bounded from L_c^∞ to BMO, T^* is bounded on L^p for all $p\in(1,\infty)$, T^* is bounded on L^p for some $p\in(1,\infty)$ and T^* is bounded from L^1 to $L^{1,\infty}$. As applications of these results, for spectral multipliers of a distinguished Laplacian on (S,d,ρ) satisfying certain Mihlin-Hörmander type condition, the authors obtain that their maximal singular integrals are bounded from L_c^∞ to BMO, from L^1 to $L^{1,\infty}$, and on L^p for all $p\in(1,\infty)$.

Keywords Exponential growth group · Maximal singular integral · Multiplier · Dyadic cube · (Local) Sharp maximal function · Fefferman–Stein type inequality

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1 Introduction

Let S be the *affine group* $\mathbb{R}^n \ltimes \mathbb{R}^+$ endowed with the *product*

$$(x,a)\cdot(x',a') = (x+ax', aa') \qquad \forall\ (x,a),(x',a')\in S.$$

We call S an *$ax+b$–group*. Clearly, $e\equiv(0,1)$ is the *unit* of S. The *inverse* of any $(x,a)\in S$, denoted by $(x,a)^{-1}$, is equal to $(-x/a,1/a)$. We endow S with the *left-invariant Riemannian metric* $ds^2\equiv a^{-2}(dx_1^2+\cdots+dx_n^2+da^2)$, which induces the following *distance function* d on $S\times S$:

$$d\big((x,a),(x',a')\big) = \cosh^{-1}\left(\frac{a^2+a'^2+|x-x'|^2}{2aa'}\right) \quad \forall\ (x,a),(x',a')\in S; \tag{1.1}$$

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see, for example, [1, formula (2.18)] or [24, pp. 119–120]. The *left Haar measure* induced by the above left-invariant Riemannian metric is $d\lambda(x,a) = a^{-n-1}dxda$. A standard calculation yields that the *group modular function* $\delta(x,a)$ is equal to a^{-n}, and hence the right Haar measure is

$$d\rho(x,a) = \delta(x,a)^{-1}d\lambda(x,a) = a^{-1}dxda. \tag{1.2}$$

Throughout the whole paper, we work on the *triple* (S,d,ρ), namely, the group S endowed with the left-invariant Riemannian metric d and the right Haar measure ρ. For all $(x,a) \in S$ and $r > 0$, we define the *ball* $B\big((x,a),r\big)$ on S by

$$B\big((x,a),r\big) \equiv \big\{(x',a') \in S: \, d\big((x,a),(x',a')\big) < r\big\}.$$

For any $p \in (0,\infty]$, let L^p be the *space* of all measurable functions f on (S,d,ρ) satisfying

$$\|f\|_{L^p} \equiv \left\{\int_S |f(x)|^p \, d\rho(x)\right\}^{1/p} < \infty,$$

with a usual modification made when $p = \infty$, and let $L^{p,\infty}$ be the *space* of all measurable functions f on (S,d,ρ) satisfying

$$\|f\|_{L^{p,\infty}} \equiv \sup_{\alpha>0}\left\{\alpha\left[\rho\left(\{x \in S: \, |f(x)| > \alpha\}\right)\right]^{1/p}\right\} < \infty.$$

Denote by L^∞_c the *set* of functions in L^∞ with compact support; and by $L^\infty_{c,0}$ the *set* of functions f in L^∞_c such that $\int_S f \, d\rho = 0$. Let L^1_{loc} be the *set* of locally integrable functions on S (with respect to the measure ρ).

It is well known that the space (S,d,ρ) is of *exponential growth at infinity*, namely,

$$\rho\Big(B\big((0,1),r\big)\Big) \approx \begin{cases} r^{n+1} & \text{if } \; r < 1, \\ e^{nr} & \text{if } \; r \geq 1; \end{cases}$$

see, for example, [24, 32]. Harmonic analysis on exponential growth groups including the space (S,d,ρ) currently attracts a lot of attention. In particular, efforts have been made to study the Hardy–Littlewood maximal function (see [11, 8]), Riesz transforms (see [27, 10, 9, 26]), spectral multipliers related to a distinguished Laplacian (see [3, 5, 16, 15, 14, 24, 32, 34]).

In 2003, Hebisch and Steger [16] introduced the notion of *Calderón–Zygmund spaces*, and developed a variant of the Calderón–Zygmund theory of singular integrals on Calderón–Zygmund spaces, which can be applied to the $ax+b$–groups. Precisely, a *space* M endowed with a *metric* d and a *Borel measure* μ is said to have the *Calderón–Zygmund property* if there exists a positive constant C such that for every $f \in L^1(\mu)$ and each $\alpha > C\|f\|_{L^1(\mu)}/\mu(M)$, there exist a decomposition $f = g + \sum_i b_i$, sets $\{Q_i\}_i \subset M$, positive numbers $\{r_i\}_i$ and points $\{x_i\}_i \subset M$ such that for all i,

(i) $\operatorname{supp} b_i \subset Q_i$ and $\int_M b_i \, d\mu = 0$,

(ii) $\|g\|_{L^\infty(\mu)} \leq C\alpha$ and $\sum_i \|b_i\|_{L^1(\mu)} \leq C\|f\|_{L^1(\mu)}$,

(iii) $Q_i \subset B(x_i, Cr_i)$ and $\sum_i \mu(Q_i^*) \leq C\|f\|_{L^1(\mu)}/\alpha$, where $Q_i^* \equiv \{x \in M: \, d(x,Q_i) < r_i\}$;

see [16, Definition 1.1]. Hebisch and Steger [16, Theorem 1.2] further proved that, for any Calderón–Zygmund space (M,d,μ), if $T\equiv\sum_{j\in\mathbb{Z}}K_j$ is bounded on $L^2(\mu)$ and if there exist positive constants $c\in(0,1)$, C, a and b such that

$$\int_M |K_j(x,y)|(1+c^j d(x,y))^a\,d\mu(x)\le C \qquad \forall\, y\in M, \tag{1.3}$$

and

$$\int_M |K_j(x,y)-K_j(x,z)|\,d\mu(x)\le C(c^j d(y,z))^b \qquad \forall\, y,z\in M, \tag{1.4}$$

then T is of weak type 1 and bounded on $L^p(\mu)$ for all $p\in(1,2]$.

Obviously, spaces of homogeneous type in the sense of Coifman and Weiss [4] enjoy the Calderón–Zygmund property. An example of Calderón–Zygmund spaces, which is not a space of homogeneous type, is (S,d,ρ) (see [16, Lemma 5.1]). In fact, each integrable function f on (S,ρ) at any level $\alpha>0$ can be decomposed into a sum $g+\sum_i b_i$ as above, where every b_i is supported on a set R_i which belongs to a suitable family $\mathcal{R}$ of sets in S: these sets are not balls because of the exponential growth of the space, but suitable "rectangles" in $\mathbb{R}^n\ltimes\mathbb{R}^+$. More precisely, for any $R\in\mathcal{R}$, there exists a positive r_R such that R is contained in a ball of radius comparable to r_R and the measures of R and its dilated set $R^*\equiv\{x\in S:\ d(x,R)<r_R\}$ are comparable. The elements in $\mathcal{R}$ are called *Calderón–Zygmund sets* on (S,d,ρ). As an application of the aforementioned Calderón–Zygmund theory of singular integrals, spectral multipliers associated with a distinguished Laplacian Δ on (S,d,ρ) which satisfy certain Mihlin-Hörmander type condition are of weak type 1 and bounded on L^p for all $p\in(1,\infty)$ [16, Theorem 2.4]. Müller and Thiele [24] re-obtained the multiplier results of [16] by considering estimates of the wave propagator associated with Δ.

A reformulation of Hebisch–Steger [16, Theorem 1.2] on (S,d,ρ), using a condition of Hörmander's type, is as follows: if T is a linear operator which is bounded on L^2 and admits a locally integrable kernel K off the diagonal satisfying that

$$\sup_{R\in\mathcal{R}}\sup_{y,\,z\in R}\int_{S\setminus R^*}|K(x,y)-K(x,z)|\,d\rho(x)<\infty, \tag{1.5}$$

then T is bounded from L^1 to $L^{1,\infty}$ and on L^p for all $p\in(1,2]$; see [34] for the details and see also [29, 13] for the Euclidean case. For the endpoint case, Vallarino [32] developed an $H^1-\mathrm{BMO}$ theory on (S,d,ρ), and proved that singular integrals whose kernels satisfy the condition (1.5) are bounded from H^1 to L^1 and from L^∞ to BMO. As an application, spectral multipliers associated with a distinguished Laplacian Δ which satisfy certain Mihlin-Hörmander type condition are bounded from H^1 to L^1 and from L^∞ to BMO (see [32, Proposition 4.2]). Moreover, Sjögren and Vallarino [27] considered H^1-L^1 boundedness of various Riesz transforms associated with Δ. In [35], the Calderón–Zygmund theory of [16] is generalized to Damek–Ricci spaces.

In this paper, we study the boundedness of maximal singular integrals on (S,d,ρ). The importance of results in this direction is well known, and comes from the fact that they imply pointwise convergence results (see, for example, [13] and, in particular, [13, Theorem

2.1.14]); see also Stein–Weiss [30, p. 60, Theorem 3.12] or Stein [28, p. 42, Theorem 4]. Recall that in the Euclidean setting, the *maximal singular integral* T^* associated with a kernel K is defined, for all suitable functions f and all x in $\mathbb{R}^n$, by

$$T^*f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y)f(y)\,dy \right|.$$

An alternative but equivalent way of expressing this *operator* T^* is

$$T^*f(x) = \sup_{B\subset\mathbb{R}^n,\, B\ni x} \left| \int_{\mathbb{R}^n\setminus 2B} K(x,y)f(y)\,dy \right|,$$

where the supremum is taken over all Euclidean balls in $\mathbb{R}^n$ containing x.

In view of this observation, in the space (S,d,ρ), we define the *maximal singular integral* T^* associated with a kernel K as

$$T^*f(x) \equiv \sup_{R\in\mathcal{R},\, R\ni x} \left| \int_{S\setminus R^*} K(x,y)f(y)\,d\rho(y) \right| \qquad \forall\, x\in S, \tag{1.6}$$

where $f\in L_c^\infty$ and the supremum is taken over all Calderón–Zygmund sets $R\in\mathcal{R}$ containing x; see Section 4 below for more details.

The main aim of this paper is to prove that, for T^*, defined as in (1.6), and associated with a kernel K that satisfies an integral size condition and Hörmander's condition (see (4.1) and (4.2) below), the following *four statements are equivalent*:

(i) T^* is bounded from L_c^∞ to BMO;
(ii) T^* is bounded on L^p for all $p\in(1,\infty)$;
(iii) T^* is bounded on L^p for some $p\in(1,\infty)$;
(iv) T^* is bounded from L^1 to $L^{1,\infty}$;

see Theorem 4.1 below. Moreover, if T is further assumed to be bounded on L^2, then the above boundedness (i) through (iv) hold for T^*; see Theorem 4.2 below.

The proof of the main results of the paper, namely, Theorems 4.1 and 4.2, are presented in Section 4. The main ingredients used in the proof are the Calderón–Zygmund property of (S,d,ρ) and certain Fefferman–Stein weak type inequalities related to the local sharp maximal functions in the sense of John [20], Strömberg [31] and Jawerth–Torchinsky [19]; see Propositions 3.1 and 3.2 below. The proof of the aforementioned Fefferman–Stein type inequalities relies on the existence of certain "dyadic" sets on (S,d,ρ), which are an analogue of the Euclidean dyadic cubes and were constructed in [22]; see Lemma 2.2 below. We remark that the proof of Theorem 4.1 invokes some ideas of [18] and Grafakos [12].

Some applications are given in Section 5. Precisely, for certain class of spectral multipliers for the distinguished Laplacian Δ we prove in Theorem 5.1 below that the corresponding maximal singular integral operators are bounded from L_c^∞ to BMO, from L^1 to $L^{1,\infty}$ and on L^p for all $p\in(1,\infty)$. The main difficulty in proving Theorem 5.1 is to show that the kernels of such spectral multipliers satisfy the integral size condition (4.1), which requires very delicate estimates (see Proposition 5.1 below).

Our paper is organized as follows. A brief recall of the geometric properties of S and the Calderón–Zygmund property is presented in Section 2. In Section 3, we establish the

Fefferman–Stein (weak) type inequalities related to the local sharp maximal functions on S. The whole Section 4 is devoted to the proof of Theorems 4.1 and 4.2, which are the main results of the paper. Finally, an application to the spectral multipliers for the Laplacian Δ is studied in Section 5.

We make some conventions on notation. Let $\mathbb{N} \equiv \{0, 1, 2, \cdots\}$ and $\mathbb{R}^+ \equiv (0, \infty)$. For any space X and any subset E of X, set $E^{\complement} \equiv X \setminus E$ and let χ_E denote the *characteristic function* of E. Denote by C a *positive constant* independent of the main parameters involved, which may vary at different occurrences. *Constants with subscripts* do not change through the whole paper. We use $f \lesssim g$ to denote $f \le Cg$. If $f \lesssim g \lesssim f$, we write $f \approx g$. For an operator T defined on the Banach space $\mathcal{A}$ and taking values in the Banach space $\mathcal{B}$, we use $\|T\|_{\mathcal{A}\to\mathcal{B}}$ to denote its *operator norm*.

2 Preliminaries

We recall the notion of Calderón–Zygmund sets, which appears in [16] and implicitly in [11]. Let $\mathcal{Q}$ be the *collection of dyadic cubes in* $\mathbb{R}^n$.

Definition 2.1. A *Calderón–Zygmund set* is a set $R = Q \times [ae^{-r}, ae^r)$, where $Q \in \mathcal{Q}$ with side length L, $a \in \mathbb{R}^+$, $r > 0$ and

$$e^2 ar \le L < e^8 ar \quad \text{if} \quad r < 1; \qquad\qquad ae^{2r} \le L < ae^{8r} \quad \text{if} \quad r \ge 1.$$

Set $a_R \equiv a$, $r_R \equiv r$ and $x_R \equiv (c_Q, a)$, where c_Q is the *center* of Q. Denote by $\mathcal{R}$ the *family of all Calderón–Zygmund sets on* S. For any $x \in S$, let $\mathcal{R}(x)$ be the *collection of all* $R \in \mathcal{R}$ *containing* x.

The following lemma presents some properties of the Calderón–Zygmund sets (see [16, 32]).

Lemma 2.1. *There exists $\kappa_0 \in [1, \infty)$ such that for all $R \in \mathcal{R}$, the following hold:*

(i) *$B(x_R, r_R) \subset R \subset B(x_R, \kappa_0 r_R)$;*

(ii) *$\rho(R^*) \le \kappa_0 \rho(R)$, where $R^* \equiv \{x \in S : d(x, R) < r_R\}$ is the dilated set of $R \in \mathcal{R}$;*

(iii) *every $R \in \mathcal{R}$ can be decomposed into mutually disjoint sets $\{R_i\}_{i=1}^k \subset \mathcal{R}$ with $k = 2$ or $k = 2^n$ such that $R = \cup_{i=1}^k R_i$ and $\rho(R_i) = \rho(R)/k$ for all $i \in \{1, \cdots, k\}$.*

Using the geometric properties of the Calderón–Zygmund sets, the authors, in [22], constructed certain "dyadic" sets on (S, d, ρ), which are analogues of the Euclidean.

Lemma 2.2. *There exists a sequence $\{\mathcal{D}_j\}_{j\in\mathbb{Z}}$ such that each $\mathcal{D}_j$ consists of pairwise disjoint Calderón–Zygmund sets, and*

(i) *for any given $j \in \mathbb{Z}$, $S = \cup_{R\in\mathcal{D}_j} R$;*

(ii) *if $\ell \le k$, $R \in \mathcal{D}_\ell$ and $R' \in \mathcal{D}_k$, then either $R \subset R'$ or $R \cap R' = \emptyset$;*

(iii) *for any given $j \in \mathbb{Z}$ and $R \in \mathcal{D}_j$, there exists a unique $R' \in \mathcal{D}_{j+1}$ such that $R \subset R'$ and $\rho(R') \le \max\{2^n, 3\}\rho(R)$;*

(iv) *for any* $j \in \mathbb{Z}$, *every* $R \in \mathcal{D}_j$ *can be decomposed into mutually disjoint sets* $\{R_i\}_{i=1}^k \subset \mathcal{D}_{j-1}$ *with* $k = 2$ *or* $k = 2^n$ *such that* $R = \cup_{i=1}^k R_i$ *and* $\rho(R)/2^n \le \rho(R_i) \le 2\rho(R)/3$ *for all* $i \in \{1, \cdots, k\}$.

From now on, we set $\mathcal{D} \equiv \{\mathcal{D}_j\}_{j\in\mathbb{Z}}$.

Hardy–Littlewood maximal functions on groups with exponential growth have been investigated in a series of works; see, for example, [8, 11, 33]. For any $f \in L^1_{\rm loc}$, we define the *Hardy–Littlewood maximal function* $\mathcal{M}f$ and the *dyadic Hardy–Littlewood maximal function* $\mathcal{M}_{\mathcal{D}}f$ respectively by the formulae

$$\mathcal{M}f(x) \equiv \sup_{R\in\mathcal{R}(x)} \frac{1}{\rho(R)} \int_R |f|\,d\rho \quad \forall\, x \in S, \tag{2.1}$$

and

$$\mathcal{M}_{\mathcal{D}}f(x) \equiv \sup_{R\in\mathcal{R}(x),\, R\in\mathcal{D}} \frac{1}{\rho(R)} \int_R |f|\,d\rho \quad \forall\, x \in S. \tag{2.2}$$

The maximal function $\mathcal{M}$ has the following boundedness properties [33].

Proposition 2.1. *$\mathcal{M}$ is bounded from L^1 to $L^{1,\infty}$, and on L^p for all $p \in (1,\infty]$.*

From (2.1), (2.2) and the differentiation theorem for integrals, it follows that

$$f(x) \le \mathcal{M}_{\mathcal{D}}f(x) \le \mathcal{M}f(x) \qquad \forall\, f \in L^1_{\rm loc},\ \text{almost every } x \in S.$$

Thus, the operator $\mathcal{M}_{\mathcal{D}}$ is bounded from L^1 to $L^{1,\infty}$ and for any $p \in (1,\infty]$,

$$\|\mathcal{M}_{\mathcal{D}}(f)\|_{L^p} \approx \|f\|_{L^p} \approx \|\mathcal{M}(f)\|_{L^p} \quad \forall\, f \in L^p.$$

It was proved in [16, Lemma 5.1] that (S, d, ρ) possesses the Calderón–Zygmund property. Indeed, the boundedness properties of $\mathcal{M}_{\mathcal{D}}$ and the differentiation theorem for integrals, together with Lemma 2.2 and a standard stopping-time argument, imply the following dyadic version of the Calderón–Zygmund property; see also [32, Proposition 2.4].

Proposition 2.2. *Let $f \in L^p$ with $p \in [1,\infty)$. For any $\alpha > 0$, there exists a sequence of disjoint sets, $\{R_i\}_i \in \mathcal{D}$, such that f can be decomposed into $f = g + \sum_i b_i$, where*

(i) $|g(x)| \le C_1\alpha$ *for almost every* $x \in S$;

(ii) *for all* i, $\operatorname{supp} b_i \subset R_i \in \mathcal{D}$ *and* $\int_S b_i\,d\rho = 0$;

(iii) *for all* i, $\alpha \le \{\frac{1}{\rho(R_i)}\int_{R_i}|f|^p\,d\rho\}^{1/p} \le C_1\alpha$;

(iv) *for all* i, $\|b_i\|_{L^p} \le C_1\alpha[\rho(R_i)]^{1/p}$,

where C_1 is a positive constant independent of α and f.

Remark 2.1. If $f \in L^\infty \cap L^p$ for some $p \in [1,\infty)$ and $f = g + \sum_i b_i$ is a Calderón–Zygmund decomposition of f obtained as in Proposition 2.2, then $\|g\|_{L^\infty} \le \|f\|_{L^\infty}$. Moreover, if $\int_S f\,d\rho = 0$, then $\int_S g\,d\rho = 0$.

3 Fefferman–Stein Type Inequalities

The local maximal functions in Euclidean spaces were introduced by John [20] and later investigated by Strömberg [31], Jawerth–Torchinsky [19] and Lerner [21]; see also [18] for the setting of spaces of homogeneous type. Following this pioneering work, we introduce the local maximal functions on (S, d, ρ).

Definition 3.1. (i) The *non-increasing rearrangement* of a measurable function f on (S, d, ρ) is defined by

$$f^*(\xi) \equiv \inf\{t > 0 : \rho(\{x \in S : |f(x)| > t\}) < \xi\} \quad \forall\, \xi \in (0, \infty).$$

(ii) Let $s \in (0, 1)$ and $\mathcal{D}$ be the family of dyadic sets. For any $f \in L^1_{\rm loc}$, its *local maximal function* $\mathcal{M}_{0,s} f$ is defined by

$$\mathcal{M}_{0,s} f(x) \equiv \sup_{R\in\mathcal{R}(x)} (f\chi_R)^* \big(s\rho(R)\big) \quad \forall\, x \in S,$$

and its *dyadic local maximal function* $\mathcal{M}^{\mathcal{D}}_{0,s} f$ is defined by

$$\mathcal{M}^{\mathcal{D}}_{0,s} f(x) \equiv \sup_{R\in\mathcal{R}(x)\cap\mathcal{D}} (f\chi_R)^* \big(s\rho(R)\big) \quad \forall\, x \in S.$$

Some properties of these local maximal operators are presented in the following lemma.

Lemma 3.1. *For any $f \in L^1_{\rm loc}$, the following hold:*

(i) *for all $s \in (0, 1)$ and $x \in S$, $\mathcal{M}^{\mathcal{D}}_{0,s} f(x) \le \mathcal{M}_{0,s} f(x)$;*

(ii) *if $0 < s_1 < s_2 < 1$, then $\mathcal{M}^{\mathcal{D}}_{0,s_2} f(x) \le \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x)$ for all $x \in S$;*

(iii) *for all $s \in (0, 1)$ and $x \in S$, $\mathcal{M}^{\mathcal{D}}_{0,s} f(x) \le s^{-1}\mathcal{M}_{\mathcal{D}} f(x)$;*

(iv) *for all $s \in (0, 1)$ and all measurable functions f_1 and f_2,*

$$\mathcal{M}^{\mathcal{D}}_{0,s}(f_1 + f_2)(x) \le \mathcal{M}^{\mathcal{D}}_{0,s/2}(f_1)(x) + \mathcal{M}^{\mathcal{D}}_{0,s/2}(f_2)(x) \qquad \forall\, x \in S.$$

(v) *Properties (ii)–(iv) hold with $\mathcal{M}^{\mathcal{D}}_{0,s}$ and $\mathcal{M}_{\mathcal{D}}$ replaced by $\mathcal{M}_{0,s}$ and $\mathcal{M}$, respectively.*

(vi) *For all $s \in (0, 1)$, $\lambda \in (0, \infty)$ and $f \in L^1_{\rm loc}$,*

$$\begin{aligned}\{x \in S : \mathcal{M}_{\mathcal{D}}(\chi_{\{|f|>\lambda\}})(x) > s\} &\subset \{x \in S : \mathcal{M}^{\mathcal{D}}_{0,s} f(x) > \lambda\}\\ &\subset \{x \in S : \mathcal{M}_{\mathcal{D}}(\chi_{\{|f|>\lambda\}})(x) \ge s\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\rho(\{x \in S : |f(x)| > \lambda\}) &\le \rho(\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,s} f(x) > \lambda\})\\ &\le \|\mathcal{M}_{\mathcal{D}}\|_{L^1\to L^{1,\infty}} s^{-1}\rho(\{x \in S : |f(x)| > \lambda\}).\end{aligned}$$

(vii) *Similarly,*

$$\begin{aligned}\{x\in S:\, \mathcal{M}(\chi_{\{|f|>\lambda\}})(x)>s\} &\subset \{x\in S:\, \mathcal{M}_{0,s}f(x)>\lambda\}\\ &\subset \{x\in S:\, \mathcal{M}(\chi_{\{|f|>\lambda\}})(x)\geq s\},\end{aligned}$$

and

$$\begin{aligned}\rho(\{x\in S:\, |f(x)|>\lambda\}) &\leq \rho(\{x\in S:\, \mathcal{M}_{0,s}f(x)>\lambda\})\\ &\leq \|\mathcal{M}\|_{L^1\to L^{1,\infty}}s^{-1}\rho(\{x\in S:\, |f(x)|>\lambda\}).\end{aligned}$$

Proof. Properties (i) and (ii) follow from the definition of $\mathcal{M}^{\mathcal{D}}_{0,s}$. For all $R\in\mathcal{R}(x)\cap\mathcal{D}$,

$$\rho(\{y\in R:\, |f(y)|>s^{-1}\mathcal{M}_{\mathcal{D}}f(x)\}) < \int_R \frac{|f(y)|}{s^{-1}\mathcal{M}_{\mathcal{D}}f(x)}\,d\rho(y)\leq s\rho(R),$$

therefore, (iii) holds. Property (iv) follows from an argument similar to the one used in [18, Lemmas 2.2], while (v) is proven as (i)-(iv). Finally, proceeding as in the proof of [18, Lemmas 2.3] yields (vi) and (vii). □

Next, we recall the notion of the median value; see [31] for the Euclidean setting.

Definition 3.2. Suppose that f is a real function in L^1_{loc} and $R\in\mathcal{R}$. A *median value* $m_f(R)$ of f over R is defined to be one of the real numbers satisfying

$$\rho(\{x\in R:\, f(x)>m_f(R)\})\leq \rho(R)/2,$$

and $\rho(\{x\in R:\, f(x)<m_f(R)\})\leq\rho(R)/2$. In the case when f is complex, define

$$m_f(R)\equiv m_{\Re(f)}(R)+im_{\Im(f)}(R),$$

where $\Re(f)$ and $\Im(f)$ denote, respectively, the *real part* and the *imaginary part* of f.

In the following lemma we show an analogue of the inequality proved by Jawerth and Torchinsky in [19, p. 238], which will be used in the proof of "good-λ" inequalities in Lemma 3.5 below.

Lemma 3.2. *Let* $f\in L^1_{\mathrm{loc}}$ *and* $R\in\mathcal{R}$. *Then*

$$|m_f(R)|\leq\sqrt{2}\inf\left\{t>0:\, \rho(\{y\in R:\, |f(y)|>t\})<\rho(R)/2\right\}. \tag{3.1}$$

Consequently, for all $s\in(0,1/2]$ *and* $R\in\mathcal{D}$,

$$|m_f(R)|\leq\sqrt{2}\inf_{x\in R}\mathcal{M}^{\mathcal{D}}_{0,1/2}f(x)\leq\sqrt{2}\inf_{x\in R}\mathcal{M}^{\mathcal{D}}_{0,s}f(x). \tag{3.2}$$

Proof. We first show (3.1). If f is real and $m_f(R) > 0$, then for all $\epsilon \in \big(0, m_f(R)\big)$,

$$\rho(\{y \in R : \, |f(y)| > m_f(R) - \epsilon\}) \geq \rho(\{y \in R : \, f(y) \geq m_f(R)\}) \geq \rho(R)/2,$$

which implies that

$$\inf \big\{t > 0 : \, \rho(\{y \in R : \, |f(y)| > t\}) < \rho(R)/2\big\} \geq m_f(R) - \epsilon.$$

Letting $\epsilon \to 0$ yields

$$\inf \big\{t > 0 : \, \rho(\{y \in R : \, |f(y)| > t\}) < \rho(R)/2\big\} \geq m_f(R). \tag{3.3}$$

If f is real and $m_f(R) < 0$, applying the above argument to $-f$ and $-m_f(R)$, we also obtain (3.3).

For any complex function f, we have

$$|m_f(R)|^2 = |m_{\Re f}(R)|^2 + |m_{\Im f}(R)|^2 \leq 2 \max\{|m_{\Re f}(R)|^2, \, |m_{\Im f}(R)|^2\}.$$

Without loss of generality, we may assume that $|m_{\Re f}(R)| \leq |m_{\Im f}(R)|$. Then, by (3.3),

$$\begin{aligned} |m_f(R)| \leq \sqrt{2} |m_{\Im f}(R)| &\leq \sqrt{2} \inf \big\{t > 0 : \, \rho(\{y \in R : \, |\Im f(y)| > t\}) < \rho(R)/2\big\} \\ &\leq \sqrt{2} \inf \big\{t > 0 : \, \rho(\{y \in R : \, |f(y)| > t\}) < \rho(R)/2\big\}. \end{aligned}$$

This proves (3.1). The inequalities (3.2) easily follow from the definition of $\mathcal{M}^{\mathcal{D}}_{0,1/2}$ and Lemma 3.1(ii). $\square$

Vallarino [32] introduced the *space* BMO of functions with bounded mean oscillation on (S, d, ρ) as follows. For any $f \in L^1_{\mathrm{loc}}$ and $R \in \mathcal{R}$, set $f_R \equiv \frac{1}{\rho(R)} \int_R f \, d\rho$; the function f is said to be in BMO if

$$\|f\|_{\mathrm{BMO}} \equiv \sup_{R \in \mathcal{R}} \frac{1}{\rho(R)} \int_R |f - f_R| \, d\rho < \infty.$$

For any $q \in (0, \infty)$ and $f \in L^1_{\mathrm{loc}}$, set

$$\|f\|_{\mathrm{BMO}_q} \equiv \sup_{R \in \mathcal{R}} \left\{ \frac{1}{\rho(R)} \int_R |f - f_R|^q \, d\rho \right\}^{1/q}.$$

It was proved in [32, Section 3] that for any $q \in (1, \infty)$, there exists a positive constant C_q such that for all $f \in L^1_{\mathrm{loc}}$,

$$\frac{1}{C_q} \|f\|_{\mathrm{BMO}_q} \leq \|f\|_{\mathrm{BMO}} \leq C_q \|f\|_{\mathrm{BMO}_q}. \tag{3.4}$$

It turns out that (3.4) also holds for $q \in (0, 1)$. This is proved in the following lemma, using some ideas of [23].

Lemma 3.3. *For any $\sigma \in (0,1)$ there exists a positive constant C_σ, which depends only on σ, such that for all $f \in L^1_{\mathrm{loc}}$,*

$$C_\sigma \|f\|_{\mathrm{BMO}} \le \|f\|_{*,\sigma} \le \|f\|_{\mathrm{BMO}_\sigma} \le \|f\|_{\mathrm{BMO}}, \tag{3.5}$$

where

$$\|f\|_{*,\sigma} \equiv \sup_{R\in\mathcal{R}} \inf_{c\in\mathbb{C}} \left\{ \frac{1}{\rho(R)} \int_R |f(x)-c|^\sigma \, d\rho(x) \right\}^{1/\sigma}.$$

Proof. The third inequality of (3.5) follows from Hölder's inequality, while the second inequality of (3.5) follows from the definitions of $\|\cdot\|_{*,\sigma}$ and $\|\cdot\|_{\mathrm{BMO}_\sigma}$.

To prove the first inequality of (3.5), we fix $f \in L^1_{\mathrm{loc}}$. Suppose that $\|f\|_{*,\sigma} < \infty$; otherwise there is nothing to prove. For any $R \in \mathcal{R}$, by the local integrability of f and the dominated convergence theorem, we have that $\{\frac{1}{\rho(R)}\int_R |f-c|^\sigma \, d\rho\}^{1/\sigma}$ is continuous with respect to c and it tends to infinity as $|c| \to \infty$. Thus, for any fixed $R \in \mathcal{R}$, there exists a certain $A_R \in \mathbb{C}$ such that

$$\inf_{c\in\mathbb{C}} \left\{ \frac{1}{\rho(R)} \int_R |f-c|^\sigma \, d\rho \right\}^{1/\sigma} = \left\{ \frac{1}{\rho(R)} \int_R |f-A_R|^\sigma \, d\rho \right\}^{1/\sigma} \le \|f\|_{*,\sigma}. \tag{3.6}$$

For all $R' \in \mathcal{R}$, by (3.6) and the fact that $|a^\sigma - b^\sigma| \le |a-b|^\sigma$ for all $a,\, b \in (0,\infty)$, we have

$$\frac{1}{\rho(R')} \int_{R'} \Big| |f-A_R|^\sigma - |A_R - A_{R'}|^\sigma \Big| \, d\rho \le \frac{1}{\rho(R')} \int_{R'} |f-A_{R'}|^\sigma \, d\rho \le \|f\|^\sigma_{*,\sigma},$$

which implies that

$$\big\| |f-A_R|^\sigma \big\|_{\mathrm{BMO}} \le 2\|f\|^\sigma_{*,\sigma}. \tag{3.7}$$

For all $a, b \in (0,\infty)$, we have $(a+b)^{1/\sigma} \le 2^{1/\sigma - 1}\left[a^{1/\sigma} + b^{1/\sigma}\right]$, which together with (3.6) yields that for all $R \in \mathcal{R}$,

$$\begin{aligned}
&\frac{1}{\rho(R)} \int_R |f-A_R| \, d\rho = \frac{1}{\rho(R)} \int_R \{|f-A_R|^\sigma\}^{1/\sigma} \, d\rho(x) \\
&\le 2^{1/\sigma-1} \left\{ \frac{1}{\rho(R)} \int_R \left| |f(x)-A_R|^\sigma - \frac{1}{\rho(R)} \int_R |f(y)-A_R|^\sigma \, d\rho(y) \right|^{1/\sigma} d\rho(x) \right. \\
&\qquad \left. + \left[\frac{1}{\rho(R)} \int_R |f(y)-A_R|^\sigma \, d\rho(y) \right]^{1/\sigma} \right\} \\
&\le 2^{1/\sigma-1} \left\{ \left(\||f-A_R|^\sigma\|_{\mathrm{BMO}_{1/\sigma}} \right)^{1/\sigma} + \|f\|_{*,\sigma} \right\}.
\end{aligned}$$

Observe that (3.4) and (3.7) imply that

$$\left(\||f-A_R|^\sigma\|_{\mathrm{BMO}_{1/\sigma}} \right)^{1/\sigma} \le \left(C_{1/\sigma} \||f-A_R|^\sigma\|_{\mathrm{BMO}} \right)^{1/\sigma} \le \left(2C_{1/\sigma}\right)^{1/\sigma} \|f\|_{*,\sigma},$$

where $C_{1/\sigma}$ is as in (3.4). Applying this estimate, we see that

$$\frac{1}{\rho(R)} \int_R |f-f_R| \, d\rho \le \frac{2}{\rho(R)} \int_R |f-A_R| \, d\rho \le 2^{1/\sigma}[(2C_{1/\sigma})^{1/\sigma} + 1]\|f\|_{*,\sigma}.$$

By taking the supremum over all $R \in \mathcal{R}$ we obtain $\|f\|_{\mathrm{BMO}} \lesssim \|f\|_{*,\sigma}$, which completes the proof. $\square$

Now we introduce the (local) sharp maximal functions on (S, d, ρ); see [13, 29, 31] for their definitions in the Euclidean setting and [18] for their definitions in spaces of homogeneous type.

Definition 3.3. (i) For any $f \in L^1_{\mathrm{loc}}$, its *sharp maximal function* $f^\sharp$ is defined by

$$f^\sharp(x) \equiv \sup_{R \in \mathcal{R}(x)} \frac{1}{\rho(R)} \int_R |f(y) - f_R| \, d\rho(y) \quad \forall\, x \in S.$$

(ii) Let $s \in (0,1)$. For any $f \in L^1_{\mathrm{loc}}$, its *local sharp maximal function* $\mathcal{M}^\sharp_{0,s} f$ is defined by

$$\mathcal{M}^{\#}_{0,s} f(x) \equiv \sup_{R \in \mathcal{R}(x)} \inf_{c \in \mathbb{C}} \big((f-c)\chi_R\big)^*\big(s\rho(R)\big) \quad \forall\, x \in S.$$

The following lemma summarizes some properties of the (local) sharp maximal functions on (S, d, ρ). The proofs are easy and hence omitted; see [13, 31, 18, 19].

Lemma 3.4. *Let $f,\, g \in L^1_{\mathrm{loc}}$. Then, for all $x \in S$,*
- (i) $f^\sharp(x) \le 2\mathcal{M}f(x)$*, where $\mathcal{M}$ is defined in* (2.1)*;*
- (ii) $|f|^\sharp(x) \le 2f^\sharp(x)$ *and* $(f+g)^\sharp(x) \le f^\sharp(x) + g^\sharp(x)$*;*
- (iii) $f^\sharp(x)/2 \le \sup_{R \in \mathcal{R}(x)} \inf_{a \in \mathbb{C}} \frac{1}{\rho(R)} \int_R |f(y) - a| \, d\rho(y) \le f^\sharp(x)$*;*
- (iv) *when $s \in (0,1)$, $\mathcal{M}^{\#}_{0,s} f(x) \le \mathcal{M}_{0,s} f(x)$ and $\mathcal{M}^{\#}_{0,s} f(x) \le s^{-1} f^{\#}(x)$, where $\mathcal{M}_{0,s}$ is defined in* (3.1)*;*
- (v) *if $s \in (0, 1/2]$ and $R \in \mathcal{R}$, then*

$$\rho(\{y \in R : \, |f(y) - m_f(R)| > 2\sqrt{2} \inf_{x \in R} \mathcal{M}^{\#}_{0,s} f(x)\}) \le s\rho(R).$$

For any $p \in (1,\infty)$ and $s \in (0,1)$, an immediate consequence of Lemma 3.4 (i) and (iv) and the L^p-boundedness of $\mathcal{M}$ is that for all $f \in L^p$,

$$\|\mathcal{M}^{\#}_{0,s} f\|_{L^p} \le 2s^{-1} \|\mathcal{M}\|_{L^p \to L^p} \|f\|_{L^p}; \tag{3.8}$$

Lemma 3.4(iv) and Lemma 3.1(vii) also imply that for all $f \in L^{p,\infty}$,

$$\|\mathcal{M}^{\#}_{0,s} f\|_{L^{p,\infty}} \le \|\mathcal{M}\|^{1/p}_{L^1 \to L^{1,\infty}} \, s^{-1/p} \|f\|_{L^{p,\infty}}. \tag{3.9}$$

Indeed, the converses of (3.8) and (3.9) hold for small s. To see this, we need the following certain kind of *"good-λ" inequalities* involving the dyadic local maximal function and the local sharp maximal function.

Lemma 3.5. *Let $0 < s_1, s_2 \le \frac{1}{2}$. Then, there exists a positive constant C_2 depending only on S such that for all $f \in L^1_{\mathrm{loc}}$ and all $\lambda > 0$,*

$$\begin{aligned} &\rho(\{x \in S : \, \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x) > 3\lambda, \, \mathcal{M}^{\#}_{0,s_2} f(x) \le \lambda/4\}) \\ &\quad \le C_2 s_1^{-1} s_2 \rho\left(\left\{x \in S : \, \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x) > \lambda\right\}\right). \end{aligned}$$

Proof. Take $\lambda > 0$ and set $\Omega_\lambda \equiv \{x \in S : \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x) > \lambda\}$. We may assume that $\rho(\Omega_\lambda) < \infty$, otherwise there is nothing to prove. For any fixed $x \in \Omega_\lambda$, there exists a "maximal dyadic cube" $R_x \in \mathcal{D}$ containing x such that

$$\inf\{t > 0 : \rho(\{y \in R_x : |f(y)| > t\}) < s_1 \rho(R_x)\} > \lambda. \tag{3.10}$$

Here "maximal dyadic cube" means that if $R' \in \mathcal{D} \cap \mathcal{R}(x)$ satisfies (3.10), then $R' \subset R_x$. Such a "maximal dyadic cube" exists since $R_x \subset \Omega_\lambda$ and $\rho(\Omega_\lambda) < \infty$. Let $\{R_j\}_{j\in I}$ be the collection of all such "maximal dyadic cubes" obtained by running x over Ω_λ. From the maximality, it follows that any two "maximal dyadic cubes" are disjoint. Moreover, $\Omega_\lambda = \cup_{j\in I} R_j$. Therefore, to show Lemma 3.5, it suffices to prove that there exists a positive constant C_2 such that for all $j \in I$,

$$\rho\left(\left\{x \in R_j : \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x) > 3\lambda,\, \mathcal{M}^{\#}_{0,s_2} f(x) \le \lambda/4\right\}\right) \le C_2 s_1^{-1} s_2 \rho(R_j). \tag{3.11}$$

Fix $j \in I$. We may assume that there exists $x_0 \in R_j$ such that $\mathcal{M}^{\#}_{0,s_2} f(x_0) \le \lambda/4$; otherwise (3.11) holds trivially. Suppose that $R_j \in \mathcal{D}_{j_0}$ for some $j_0 \in \mathbb{Z}$. Using Lemma 2.2(iii), we take $\widetilde{R}_j$ to be the unique Calderón–Zygmund set in $\mathcal{D}_{j_0+1}$ that contains R_j. Then, $\rho(\widetilde{R}_j) \le \max\{2^n, 3\}\rho(R_j)$. By the maximality of R_j and (3.10), we have

$$\inf\left\{t > 0 : \rho(\{y \in \widetilde{R}_j : |f(y)| > t\}) < s_1 \rho(\widetilde{R}_j)\right\} \le \lambda. \tag{3.12}$$

From this and Lemma 3.2 together with the hypothesis $s_1 \le 1/2$, it follows that

$$|m_f(\widetilde{R}_j)| \le \sqrt{2} \inf\{t > 0 : \rho(\{y \in \widetilde{R}_j : |f(y)| > t\}) < \rho(\widetilde{R}_j)/2\} \le \sqrt{2}\lambda.$$

Thus,

$$\mathcal{M}^{\mathcal{D}}_{0,s_1/2}(m_f(\widetilde{R}_j)\chi_{\widetilde{R}_j})(x) \le |m_f(\widetilde{R}_j)| \le \sqrt{2}\lambda.$$

If $\mathcal{M}^{\mathcal{D}}_{0,s_1}(f)(x) > 3\lambda$, then by (3.12) and the definition of $\widetilde{R}_j$, we have

$$\mathcal{M}^{\mathcal{D}}_{0,s_1}(f\chi_{\widetilde{R}_j})(x) = \mathcal{M}^{\mathcal{D}}_{0,s_1}(f)(x) > 3\lambda.$$

So applying Lemma 3.1(iv) yields that for all $x \in R_j$ satisfying $\mathcal{M}^{\mathcal{D}}_{0,s_1}(f)(x) > 3\lambda$,

$$\begin{aligned}
&\mathcal{M}^{\mathcal{D}}_{0,s_1/2}\left((f - m_f(\widetilde{R}_j))\chi_{\widetilde{R}_j}\right)(x)\\
&\quad \ge \mathcal{M}^{\mathcal{D}}_{0,s_1}\left(f\chi_{\widetilde{R}_j}\right)(x) - \mathcal{M}^{\mathcal{D}}_{0,s_1/2}\left(m_f(\widetilde{R}_j)\chi_{\widetilde{R}_j}\right)(x) > \lambda.
\end{aligned} \tag{3.13}$$

By using (3.13), Lemma 3.1(vi), $\mathcal{M}^{\#}_{0,s_2} f(x_0) \le \lambda/4$ and Lemma 3.4(v), we obtain

$$\begin{aligned}
&\rho\left(\left\{x \in R_j : \mathcal{M}^{\mathcal{D}}_{0,s_1} f(x) > 3\lambda,\, \mathcal{M}^{\#}_{0,s_2} f(x) \le \lambda/4\right\}\right)\\
&\quad \le \rho\left(\left\{x \in R_j : \mathcal{M}^{\mathcal{D}}_{0,s_1/2}\left((f - m_f(\widetilde{R}_j))\chi_{\widetilde{R}_j}\right)(x) > \lambda\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le 2\|\mathcal{M}\|_{L^1\to L^{1,\infty}} s_1^{-1} \rho\left(\left\{x \in \widetilde{R}_j : |f(x) - m_f(\widetilde{R}_j)| > \lambda\right\}\right) \\
&\le 2\|\mathcal{M}\|_{L^1\to L^{1,\infty}} s_1^{-1} \rho\left(\left\{x \in \widetilde{R}_j : |f(x) - m_f(\widetilde{R}_j)| > 2\sqrt{2} \inf_{z\in \widetilde{R}_j} \mathcal{M}^{\#}_{0,s_2} f(z)\right\}\right) \\
&\le 2\|\mathcal{M}\|_{L^1\to L^{1,\infty}} s_1^{-1} s_2 \rho(\widetilde{R}_j).
\end{aligned}
$$

Hence, (3.11) holds with $C_2 \equiv 2\max\{2^n, 3\}\|\mathcal{M}\|_{L^1\to L^{1,\infty}}$. This finishes the proof. □

Applying the previous"good-λ" inequality, we prove the following Fefferman–Stein type inequality.

Proposition 3.1. *Let $p_0 \in (0,\infty)$ and C_2 be as in Lemma 3.5. Then, for any $p \in (p_0,\infty)$ and $s \in (0, 1/2]$ satisfying $s < (2^2 3^p C_2)^{-1}$, there exists a positive constant C such that*

$$\|f\|_{L^p} \le C\|\mathcal{M}^{\#}_{0,s} f\|_{L^p} \quad \forall\ f \in L^{p_0,\infty}.$$

Proof. Fix $f \in L^{p_0,\infty}$. For any $N \in \mathbb{N}$, set

$$\mathrm{I_N} \equiv \int_0^{3N} \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2} f(x) > \lambda\right\}\right) p\lambda^{p-1} d\lambda.$$

Then, Lemma 3.1(vi) and $p > p_0$ imply that $\mathrm{I_N} < \infty$. Applying Lemma 3.5 yields that

$$
\begin{aligned}
\mathrm{I_N} &= 3^p \int_0^N p\lambda^{p-1} \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2} f(x) > 3\lambda\right\}\right) d\lambda \\
&\le 3^p \left[\int_0^N p\lambda^{p-1} \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2} f(x) > 3\lambda, \mathcal{M}^{\sharp}_{0,s} f(x) \le \lambda/4\right\}\right) d\lambda \right. \\
&\quad \left. + \int_0^N p\lambda^{p-1} \rho\left(\left\{x \in S : \mathcal{M}^{\sharp}_{0,s} f(x) > \lambda/4\right\}\right) d\lambda\right] \\
&\le 3^p 2C_2 s \int_0^N p\lambda^{p-1} \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2} f(x) > \lambda\right\}\right) d\lambda + 3^p 4^p \|\mathcal{M}^{\sharp}_{0,s} f\|^p_{L^p} \\
&\le s 3^p 2C_2 \mathrm{I}_N + 3^p 4^p \|\mathcal{M}^{\sharp}_{0,s} f\|^p_{L^p}.
\end{aligned}
$$

Since $s < (3^p 2^2 C_2)^{-1}$, we have $\mathrm{I}_N \lesssim \|\mathcal{M}^{\sharp}_{0,s} f\|^p_{L^p}$. Letting $N \to \infty$ and using Lemma 3.1(vi), we obtain $\|f\|_{L^p} \le \|\mathcal{M}^{\mathcal{D}}_{0,1/2} f\|_{L^p} \lesssim \|\mathcal{M}^{\#}_{0,s} f\|_{L^p}$, which completes the proof. □

The corresponding weak-type Fefferman–Stein inequality is the following.

Proposition 3.2. *Let $p_0 \in (0,\infty)$ and C_2 be as in Lemma 3.5. Then, for any $p \in [p_0,\infty)$ and $s \in (0, 1/2]$ satisfying $s < (2^2 3^p C_2)^{-1}$, there exists a positive constant C such that*

$$\|f\|_{L^{p,\infty}} \le C\|\mathcal{M}^{\sharp}_{0,s} f\|_{L^{p,\infty}} \qquad \forall\ f \in L^{p_0,\infty}.$$

Proof. For any $N \in \mathbb{N}$, set $\mathrm{I_N} \equiv \sup_{0<\lambda<3N} \lambda^{\mathrm{p}} \rho(\{\mathrm{x} \in \mathrm{S} : \mathcal{M}^{\mathcal{D}}_{0,1/2}\mathrm{f}(\mathrm{x}) > \lambda\})$. Combining Lemma 3.1(vi) with $f \in L^{p_0,\infty}$ and $p \geq p_0$ implies that $\mathrm{I_N} < \infty$. Thus, by Lemma 3.5,

$$\begin{aligned}
\mathrm{I_N} &= 3^p \sup_{0<\lambda<N} \lambda^p \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2}f(x) > 3\lambda\right\}\right) \\
&\leq 3^p \sup_{0<\lambda<N} \lambda^p \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2}f(x) > 3\lambda,\, \mathcal{M}^{\sharp}_{0,s}f(x) \leq \lambda/4\right\}\right) \\
&\quad +3^p \sup_{0<\lambda<N} \lambda^p \rho\left(\left\{x \in S : \mathcal{M}^{\sharp}_{0,s}f(x) > \lambda/4\right\}\right) \\
&\leq 2sC_2 3^p \sup_{0<\lambda<N} \lambda^p \rho\left(\left\{x \in S : \mathcal{M}^{\mathcal{D}}_{0,1/2}f(x) > \lambda\right\}\right) + 3^p 4^p \|\mathcal{M}^{\sharp}_{0,s}f\|^p_{L^{p,\infty}} \\
&\leq s2 3^p C_2 \mathrm{I}_N + 3^p 4^p \|\mathcal{M}^{\sharp}_{0,s}f\|^p_{L^{p,\infty}}.
\end{aligned}$$

Since $\mathrm{I_N}$ is finite, the assumption $s < (2^2 3^p C_2)^{-1}$ implies that $\mathrm{I_N} \lesssim \|\mathcal{M}^{\sharp}_{0,\mathrm{s}}\mathrm{f}\|^{\mathrm{p}}_{\mathrm{L^{p,\infty}}}$. Letting $N \to \infty$ and using Lemma 3.1(vi) yield the desired conclusion. □

Remark 3.1. A Fefferman–Stein inequality involving the sharp maximal function also holds. More precisely, let p_0 be in $(1, \infty)$. Then for any $p \in (p_0, \infty)$, by Lemma 3.4(iv) and Proposition 3.1,

$$\|f\|_{L^p} \leq C\|f^{\sharp}\|_{L^p} \qquad \forall\ f \in L^{p_0,\infty}. \tag{3.14}$$

As we shall see, such a Fefferman–Stein inequality is not enough for the proof of Theorem 4.1 below. This explains why we studied Fefferman–Stein type inequalities related to the local sharp maximal function as in Proposition 3.2, which is crucial in the proof of Theorem 4.1. However, (3.14) would be enough to give a direct proof of Theorem 4.2 below.

4 Maximal Singular Integrals

In this section, we consider the boundedness of maximal singular integrals whose kernels satisfy some integral size condition and Hörmander's condition.

Assume that K is a locally integrable function on $(S \times S) \setminus \{(x, x) : x \in S\}$ such that

$$\sup_{y\in S} \sup_{r>0} \int_{r<d(x,y)\leq 2r} \left[|K(x, y)| + |K(y, x)|\right] d\rho(x) \equiv \nu_1 < \infty, \tag{4.1}$$

and

$$\sup_{R\in\mathcal{R}} \sup_{y,\,y'\in R} \int_{(R^*)^{\complement}} \left[|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)|\right]\, d\rho(x) \equiv \nu_2 < \infty, \tag{4.2}$$

where, for any R in $\mathcal{R}$, $R^* \equiv \{x \in S : d(x, R) < r_R\}$ and $(R^*)^{\complement} \equiv S \setminus R^*$.

Let T be the *linear operator associated with a kernel* K satisfying (4.1) and (4.2); in particular, for all $f \in L^\infty_c$ and $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_S K(x, y)f(y)\, d\rho(y).$$

We define the *maximal singular integral operator* T^* by

$$T^*f(x) \equiv \sup_{R\in\mathcal{R}(x)} |T_Rf(x)| \qquad \forall\ f\in L^\infty_c,\ \forall\ x\in S, \tag{4.3}$$

where T_R is the *truncated operator* defined by

$$T_Rf(x) \equiv \int_{(R^*)^\complement} K(x,y)f(y)\,d\rho(y) \qquad \forall\ x\in S,\ \forall\ R\in\mathcal{R}(x).$$

The main result concerning such maximal singular integrals is the following.

Theorem 4.1. *Suppose that T^* is the maximal singular integral operator as in* (4.3) *associated with a kernel K satisfying* (4.1) *and* (4.2)*. The following statements are equivalent:*

(i) *T^* is bounded from L^∞_c to* BMO *;*

(ii) *T^* is bounded on L^p for all $p\in(1,\infty)$;*

(iii) *T^* is bounded on L^p for some $p\in(1,\infty)$;*

(iv) *T^* is bounded from L^1 to $L^{1,\infty}$.*

To prove Theorem 4.1, we first establish the following lemma by following some ideas used by Grafakos [12, Lemma 1].

Lemma 4.1. *Let T^* and K be as in Theorem 4.1. Suppose that $p\in[1,\infty)$, $\lambda>0$, $b\equiv\sum_{i\in I}b_i$ and $\{R_i\}_{i\in I}\subset\mathcal{R}$ are such that for a fixed positive constant C_3 and all $i\in I$, $\operatorname{supp} b_i\subset R_i$, $\int_S b_i\,d\rho=0$, $\|b_i\|_{L^p}\le C_3\lambda[\rho(R_i)]^{1/p}$, and $\{R_i\}_{i\in I}$ are pairwise disjoint. Then*

$$\rho\left(\left\{x\notin\bigcup_{i\in I}R_i^*:\ T^*b(x)>(C_{\kappa_0}C_3\nu_1+3)\lambda\right\}\right)\le C_3\left[\nu_1\widetilde{C}_{\kappa_0}+3\nu_2\right]\sum_{i\in I}\rho(R_i), \tag{4.4}$$

where κ_0 is the constant which appears in Lemma 2.1, $C_{\kappa_0}\equiv 3+\log_2(\kappa_0+1)$ and $\widetilde{C}_{\kappa_0}\equiv 2+\log_2[(4\kappa_0+3)\kappa_0]$.

Proof. For any fixed $x\notin\bigcup_{i\in I}R_i^*$ and $R\in\mathcal{R}(x)$, we set $\mathrm{I}_1(x,R)\equiv\{i\in I:\ R_i\subset R^*\}$, $\mathrm{I}_2(x,R)\equiv\{i\in I:\ R_i\cap R^*=\emptyset\}$ and $\mathrm{I}_3(x,R)\equiv\{i\in I:\ R_i\cap R^*\neq\emptyset,\ R_i\cap(R^*)^\complement\neq\emptyset\}$. Then, by the definition of T_R,

$$|T_Rb(x)|\le\left|T_R\left(\sum_{i\in\mathrm{I}_2(x,R)}b_i\right)(x)\right|+\left|T_R\left(\sum_{i\in\mathrm{I}_3(x,R)}b_i\right)(x)\right|. \tag{4.5}$$

Denote by x_i the center of R_i. By $\int_S b_i\,d\rho=0$ and $\operatorname{supp} b_i\subset R_i$, we obtain

$$\left|T_R\left(\sum_{i\in\mathrm{I}_2(x,R)}b_i\right)(x)\right|=\left|\sum_{i\in\mathrm{I}_2(x,R)}\int_{R_i}[K(x,y)-K(x,x_i)]b_i(y)\,d\rho(y)\right|$$

$$\leq \sum_{i\in I}\int_{R_i}|K(x,y)-K(x,x_i)||b_i(y)|\,d\rho(y)\equiv \mathrm{Z}_1(x).$$

To estimate the second term in the right-hand side of (4.5), for any $i\in \mathrm{I}_3(x,R)$, we set

$$c_i(R)\equiv \frac{1}{\rho(R_i)}\int_{R_i} b_i(y)\chi_{_{(R^*)^\complement}}(y)\,d\rho(y).$$

Notice that

$$\begin{aligned}\left|T_R\left(\sum_{i\in \mathrm{I}_3(x,R)} b_i\right)(x)\right| &= \left|\sum_{i\in \mathrm{I}_3(x,R)}\int_{R_i} K(x,y)b_i(y)\chi_{_{(R^*)^\complement}}(y)\,d\rho(y)\right|\\ &\leq \left|\sum_{i\in \mathrm{I}_3(x,R)}\int_{R_i} K(x,y)\left[b_i(y)\chi_{_{(R^*)^\complement}}(y)-c_i(R)\right]\,d\rho(y)\right|\\ &\quad+\left|\sum_{i\in \mathrm{I}_3(x,R)}\int_{R_i} K(x,y)c_i(R)\,d\rho(y)\right|\equiv \mathrm{J}_1(x)+\mathrm{J}_2(x). \quad (4.6)\end{aligned}$$

For all $i\in \mathrm{I}_3(x,R)$, by the hypothesis $\|b_i\|_{L^p}\leq C_3\lambda[\rho(R_i)]^{1/p}$, we have $|c_i(R)|\leq C_3\lambda$. From this, it follows that

$$\begin{aligned}\mathrm{J}_1(x) &= \left|\sum_{i\in \mathrm{I}_3(x,R)}\int_{R_i}[K(x,y)-K(x,x_i)]\left[b_i(y)\chi_{_{(R^*)^\complement}}(y)-c_i(R)\right]\,d\rho(y)\right|\\ &\leq \mathrm{Z}_1(x)+C_3\lambda\sum_{i\in I}\int_{R_i}|K(x,y)-K(x,x_i)|\,d\rho(y)\equiv \mathrm{Z}_1(x)+\mathrm{Z}_2(x).\end{aligned}$$

To estimate $\mathrm{J}_2(x)$, set $\mathrm{I}_3^{(1)}(x,R)\equiv\{i\in \mathrm{I}_3(x,R):\, r_R\geq 4\kappa_0 r_{R_i}\}$ and

$$\mathrm{I}_3^{(2)}(x,R)\equiv\{i\in \mathrm{I}_3(x,R):\, r_R< 4\kappa_0 r_{R_i}\}.$$

For any $i\in \mathrm{I}_3(x,R)$, assume that $w_i\in R_i\cap R^*$ and $u_i\in R_i\cap (R^*)^\complement$. If $i\in \mathrm{I}_3^{(1)}(x,R)$, by the assumption that $x\in R$ and Lemma 2.1(i), we obtain that for all $y\in R_i$,

$$d(x,y)\leq d(x,w_i)+d(w_i,y)<(2\kappa_0+1)r_R+2\kappa_0 r_{R_i}<2(\kappa_0+1)r_R$$

and

$$d(x,y)\geq d(x,u_i)-d(u_i,y)>r_R-2\kappa_0 r_{R_i}\geq r_R/2.$$

These facts together with the pairwise disjointness of $\{R_i\}_{i\in I}$ yield

$$\left|\sum_{i\in \mathrm{I}_3^{(1)}(x,R)}\int_{R_i} K(x,y)c_i(R)\,d\rho(y)\right|\leq C_3\lambda\sum_{i\in \mathrm{I}_3^{(1)}(x,R)}\int_{R_i}|K(x,y)|\,d\rho(y)$$

$$
\begin{aligned}
&\le C_3\lambda \int_{r_R/2<d(x,y)\le 2(\kappa_0+1)r_R} |K(x,y)|\,d\rho(y)\\
&< [3+\log_2(\kappa_0+1)]C_3\nu_1\lambda.
\end{aligned}
$$

If $i \in \mathrm{I}_3^{(2)}(x,R)$ and since $x \in R\setminus \cup_{i\in I} R_i^*$, for all $y \in R_i$,

$$
r_{R_i} < d(x,y) \le d(x,w_i) + d(w_i,y) \le (2\kappa_0+1)r_R + 2\kappa_0 r_{R_i} < 2(4\kappa_0+3)\kappa_0 r_{R_i},
$$

which implies that

$$
\begin{aligned}
\left|\sum_{i\in \mathrm{I}_3^{(2)}(x,R)} \int_{R_i} K(x,y)c_i(R)\,d\rho(y)\right| &\le C_3\lambda \sum_{i\in I}\int_{r_{R_i}<d(x,y)\le 2(4\kappa_0+3)\kappa_0 r_{R_i}} |K(x,y)|\chi_{R_i}(y)\,d\rho(y)\\
&\equiv \mathrm{Z}_3(x).
\end{aligned}
$$

Therefore, $\mathrm{J}_2(x) \le [3+\log_2(\kappa_0+1)]C_3\nu_1\lambda + \mathrm{Z}_3(x)$.

Set $C_{\kappa_0} \equiv 3+\log_2(\kappa_0+1)$. The estimates of $\mathrm{J}_1(x)$ and $\mathrm{J}_2(x)$ together with (4.5) and (4.6) imply that for all $x \notin \cup_{i\in I} R_i^*$,

$$
T^*b(x) \le 2\mathrm{Z}_1(x) + \mathrm{Z}_2(x) + C_{\kappa_0}C_3\nu_1\lambda + \mathrm{Z}_3(x). \tag{4.7}
$$

By this, (4.1) and (4.2), we obtain

$$
\begin{aligned}
&\rho\left(\left\{x\notin \bigcup_{i\in I} R_i^* : T^*b(x) > (C_{\kappa_0}C_3\nu_1+3)\lambda\right\}\right)\\
&\le \rho\left(\left\{x\notin \bigcup_{i\in I} R_i^* : \mathrm{Z}_1(x) > \lambda/2\right\}\right) + \rho\left(\left\{x\notin \bigcup_{i\in I} R_i^* : \mathrm{Z}_2(x) > \lambda\right\}\right)\\
&\quad +\rho\left(\left\{x\notin \bigcup_{i\in I} R_i^* : \mathrm{Z}_3(x) > \lambda\right\}\right)\\
&\le \frac{2}{\lambda}\sum_{i\in I}\int_{S\setminus R_i^*}\int_{R_i} |K(x,y)-K(x,x_i)||b_i(y)|\,d\rho(y)\,d\rho(x)\\
&\quad +C_3\sum_{i\in I}\int_{S\setminus R_i^*}\int_{R_i} |K(x,y)-K(x,x_i)|\,d\rho(y)\,d\rho(x)\\
&\quad +C_3\sum_{i\in I}\int_{S\setminus R_i^*}\int_{r_{R_i}<d(x,y)\le 2(4\kappa_0+3)\kappa_0 r_{R_i}} |K(x,y)|\chi_{R_i}(y)\,d\rho(y)\,d\rho(x)\\
&\le \frac{2\nu_2}{\lambda}\sum_{i\in I}\|b_i\|_{L^1} + C_3\nu_2\sum_{i\in I}\rho(R_i) + C_3\nu_1\Big(2+\log_2[(4\kappa_0+3)\kappa_0]\Big)\sum_{i\in I}\rho(R_i).
\end{aligned} \tag{4.8}
$$

The hypothesis $\|b_i\|_{L^p} \le C_3\lambda[\rho(R_i)]^{1/p}$ implies that

$$
\sum_{i\in I}\|b_i\|_{L^1} \le \sum_{i\in I}\|b_i\|_{L^p}[\rho(R_i)]^{1-1/p} \le C_3\lambda\sum_{i\in I}\rho(R_i).
$$

Combining all these facts yields (4.4). This finishes the proof. $\square$

Lemma 4.2. *Let T^* and K be as in Theorem 4.1. If T^* is bounded from L^∞_c to* BMO*, then $T^* f \in L^{1,\infty}$ for all $f \in L^\infty_{c,0}$.*

Proof. Without loss of generality, we may assume that $\operatorname{supp} f \subset R_0 \in \mathcal{R}$. Since $f \in L^\infty_{c,0}$, we have that $T^* f \in$ BMO, which implies the local integrability of $T^* f$ and

$$\sup_{\alpha>0} \alpha\rho\left(\{x \in (R_0)^* : T^* f(x) > \alpha\}\right) \le \|(T^* f)\chi_{(R_0)^*}\|_{L^1} < \infty.$$

Therefore, it suffices to prove that

$$\sup_{\alpha>0} \alpha\rho\left(\{x \notin (R_0)^* : T^* f(x) > \alpha\}\right) < \infty. \tag{4.9}$$

To this end, fix $x \notin (R_0)^*$ and $R \in \mathcal{R}(x)$. If $R_0 \subset R^*$, then $T_R f(x) = 0$. If $R_0 \cap R^* = \emptyset$, then we denote by x_0 the center of R_0 and use the fact that $\int_S f\, d\rho = 0$ to obtain

$$|T_R f(x)| \le \int_{R_0} |K(x,y) - K(x,x_0)||f(y)|\, d\rho(y) \equiv \mathrm{Y}_1(x).$$

If $R_0 \cap R^* \ne \emptyset$ and $R_0 \cap (R^*)^{\complement} \ne \emptyset$, we take a Calderón–Zygmund decomposition of f at level α and write $f = g^\alpha + b^\alpha$ with $b^\alpha \equiv \sum_{i\in I_\alpha} b_i^\alpha$, $\|g^\alpha\|_{L^\infty} \le C_1\alpha$, $\operatorname{supp} b_i^\alpha \subset R_i^\alpha$, $\{R_i^\alpha\}_{i\in I_\alpha} \subset \mathcal{R}$ are mutually disjoint, $\int_S b_i^\alpha\, d\rho = 0$, $\alpha \le \frac{1}{\rho(R_i^\alpha)} \int_{R_i^\alpha} |f|\, d\rho \le C_1\alpha$ and $\|b_i^\alpha\|_{L^1} \le C_1\alpha\rho(R_i^\alpha)$, where I_α is a certain index set and C_1 is the constant which appears in Proposition 2.2. Then

$$\begin{aligned}
|T_R f(x)| &= \left|\int_{R_0\cap(R^*)^{\complement}} K(x,y) f(y)\, d\rho(y)\right| \\
&\le \left|\int_{R_0\cap(R^*)^{\complement}} K(x,y) g^\alpha(y)\, d\rho(y)\right| + \sum_{i\in I_\alpha} \left|\int_{R_0\cap(R^*)^{\complement}} K(x,y) b_i^\alpha(y)\, d\rho(y)\right| \\
&\equiv \mathrm{Y}_2(x) + \mathrm{Y}_3(x).
\end{aligned}$$

Observe that if $R_0 \cap R^* \ne \emptyset$ and $R_0 \cap (R^*)^{\complement} \ne \emptyset$, then for all $y \in R_0 \cap (R^*)^{\complement}$,

$$\frac{r_R + r_{R_0}}{2} \le d(x,y) \le d(x,y_1) + d(y_1,y) \le (1+2\kappa_0) r_R + 2\kappa_0 r_{R_0} < (2\kappa_0+1)(r_R + r_{R_0}),$$

where y_1 is some fixed point in $R_0 \cap R^*$. Thus, for all $x \notin (R_0)^*$,

$$\mathrm{Y}_2(x) \le \int_{\frac{r_R + r_{R_0}}{2} \le d(x,y) < (2\kappa_0+1)(r_R + r_{R_0})} |K(x,y)||g^\alpha(y)|\, d\rho(y) \le \nu_1 C_1 \alpha \left[3 + \log_2(2\kappa_0+1)\right].$$

By proceeding as in the proof of Lemma 4.1, we obtain that for all $x \notin \cup_{i\in I_\alpha} (R_i^\alpha)^*$,

$$\mathrm{Y}_3(x) \le 2\mathrm{Z}_1'(x) + \mathrm{Z}_2'(x) + C_{\kappa_0} C_1 \nu_1 \alpha + \mathrm{Z}_3'(x),$$

where C_{κ_0} is as in Lemma 4.1, and

$$\mathrm{Z}_1'(x) \equiv \sum_{i\in I_\alpha} \int_{R_i^\alpha} |K(x,y) - K(x,x_i)||b_i^\alpha(y)|\, d\rho(y),$$

$$\mathrm{Z}_2'(x) \equiv C_1\alpha \sum_{i\in I_\alpha} \int_{R_i^\alpha} |K(x,y)-K(x,x_i)|\,d\rho(y),$$

$$\mathrm{Z}_3'(x) \equiv C_1\alpha \sum_{i\in I_\alpha} \int_{r_{R_i^\alpha}<d(x,y)\le 2(4\kappa_0+3)\kappa_0 r_{R_i^\alpha}} |K(x,y)|\chi_{R_i^\alpha}(y)\,d\rho(y).$$

Combining all these estimates, we obtain that for all $x \notin \cup_{i\in I_\alpha}(R_i^\alpha)^*$

$$T^*f(x) \le \mathrm{Y}_1(x) + \nu_1 C_1\alpha[3+\log_2(2\kappa_0+1)] + 2\mathrm{Z}_1'(x) + \mathrm{Z}_2'(x) + C_{\kappa_0}C_1\nu_1\alpha + \mathrm{Z}_3'(x),$$

and hence for all $\alpha > 0$,

$$\begin{aligned} &\rho\big(\{x\notin (R_0)^* : T^*f(x) > (C_{\kappa_0}\nu_1 C_1 + 4 + \nu_1 C_1[3+\log_2(2\kappa_0+1)])\alpha\}\big)\\ &\quad \le \rho\big(\cup_{i\in I_\alpha}(R_i^\alpha)^*\big) + \rho\big(\{x\notin (R_0)^* : \mathrm{Y}_1(x) > \alpha\}\big)\\ &\qquad + \rho\big(\{x\notin \cup_{i\in I_\alpha}(R_i^\alpha)^* : 2\mathrm{Z}_1'(x)+\mathrm{Z}_2'(x)+\mathrm{Z}_3'(x) > 3\alpha\}\big). \end{aligned}$$

By Lemma 2.1(ii), the condition $\alpha \le \frac{1}{\rho(R_i^\alpha)}\int_{R_i^\alpha}|f|\,d\rho$ and the pairwise disjointness of $\{R_i^\alpha\}_{i\in I_\alpha}$, we have $\rho\left(\cup_{i\in I_\alpha}(R_i^\alpha)^*\right) \le \kappa_0\sum_{i\in I_\alpha}\rho(R_i^\alpha) \le \kappa_0\alpha^{-1}\|f\|_{L^1}$, where κ_0 is the constant which appears in Lemma 2.1. For the second term, applying (4.2) yields

$$\begin{aligned} \rho\left(\{x\notin (R_0)^* : \mathrm{Y}_1(x) > \alpha\}\right) &\le \alpha^{-1}\int_{x\notin(R_0)^*}\int_{R_0}|K(x,y)-K(x,x_0)||f(y)|\,d\rho(y)\,d\rho(x)\\ &\le \alpha^{-1}\nu_2\|f\|_{L^1}. \end{aligned}$$

Finally, an argument similar to (4.8) yields that

$$\rho\left(\left\{x\notin\cup_{i\in I_\alpha}(R_i^\alpha)^* : 2\mathrm{Z}_1'(x)+\mathrm{Z}_2'(x)+\mathrm{Z}_3'(x) > 3\alpha\right\}\right) \le C_1\left[\nu_1\widetilde{C}_{\kappa_0}+3\nu_2\right]\|f\|_{L^1},$$

where $\widetilde{C}_{\kappa_0}$ is the constant which appears in Lemma 4.1. Combining all these estimates gives (4.9). Hence, $T^*f \in L^{1,\infty}$. $\square$

Proof of Theorem 4.1. To show that (i) implies (ii), by the Marcinkiewicz interpolation theorem (see [13, Theorem 1.4.19]) and the fact that $L^\infty_{c,0}$ is dense in L^q when $q\in(1,\infty)$ (see [32, Lemma 5.3]), it suffices to show that for any $p\in(1,\infty)$ and all $f\in L^\infty_{c,0}$,

$$\|T^*f\|_{L^{p,\infty}} \lesssim \|f\|_{L^p}. \tag{4.10}$$

To prove (4.10), we fix $p\in(1,\infty)$ and $f\in L^\infty_{c,0}$. By applying Proposition 2.2 and Remark 2.1, we know that for any given $\alpha>0$, there exist a positive constant C_1 and a sequence of pairwise disjoint sets $\{R_i^\alpha\}_{i\in I_\alpha}\subset\mathcal{R}$ such that f is decomposed into $f = g^\alpha + b^\alpha = g^\alpha + \sum_{i\in I_\alpha} b_i^\alpha$ such that

(a) $|g^\alpha(x)| \le C_1\alpha$ for almost all $x\in S$ and $\|g^\alpha\|_{L^\infty} \le \|f\|_{L^\infty}$;

(b) for all $i\in I_\alpha$, $\operatorname{supp} b_i^\alpha \subset R_i^\alpha \in \mathcal{R}$ and $\int_S b_i^\alpha\,d\rho = 0$;

(c) for all $i\in I_\alpha$, $\alpha \le \{\frac{1}{\rho(R_i^\alpha)}\int_{R_i^\alpha}|f|^p\,d\rho\}^{1/p} \le C_1\alpha$;

(d) for all $i\in I_\alpha$, $\|b_i^\alpha\|_{L^p} \le C_1\alpha[\rho(R_i^\alpha)]^{1/p}$,

where I_α is an index set and C_1 is the constant which appears in Proposition 2.2. By proceeding as in the proof of [18, (3.5)], we obtain that for all $s \in (0,1)$, $\alpha > 0$, and $x \in S$,

$$\mathcal{M}^\sharp_{0,s}(T^*f)(x) = \mathcal{M}^\sharp_{0,s}\big(T^*(g^\alpha + b^\alpha)\big)(x) \le \mathcal{M}^\sharp_{0,s/2}(T^*g^\alpha)(x) + \mathcal{M}_{0,s/2}(T^*b^\alpha)(x);$$

therefore, with C_{κ_0} as in Lemma 4.1 and $C_4 \equiv \|T^*\|_{L^\infty_c \to \mathrm{BMO}}$,

$$\begin{aligned}
&\rho(\{x \in S : \mathcal{M}^\sharp_{0,s}(T^*f)(x) > (2C_4C_1s^{-1} + C_{\kappa_0}C_1\nu_1 + 3)\alpha\}) \\
&\quad \le \rho(\{x \in S : \mathcal{M}^\sharp_{0,s/2}(T^*g^\alpha)(x) > 2C_4C_1s^{-1}\alpha\}) \\
&\qquad + \rho(\{x \in S : \mathcal{M}_{0,s/2}(T^*b^\alpha)(x) > (C_{\kappa_0}C_1\nu_1 + 3)\alpha\}) \equiv \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By (i), Lemma 3.4(iv) and Property (a), we obtain

$$\|\mathcal{M}^\sharp_{0,s/2}(T^*g^\alpha)\|_{L^\infty} \le 2s^{-1}\|(T^*g^\alpha)^\sharp\|_{L^\infty} \le 2s^{-1}\|T^*\|_{L^\infty_c \to \mathrm{BMO}}\|g^\alpha\|_{L^\infty} \le 2C_4C_1s^{-1}\alpha,$$

which implies that $\mathrm{I} = 0$. By Lemma 3.1(vii), the term II can be estimated by

$$\mathrm{II} \le 2\|\mathcal{M}\|_{L^1 \to L^{1,\infty}} s^{-1} \rho\left(\{x \in S : T^*b^\alpha(x) > (C_{\kappa_0}C_1\nu_1 + 3)\alpha\}\right),$$

then, applying Lemma 4.1, Lemma 2.1(ii) and Property (c) yields that

$$\begin{aligned}
\mathrm{II} &\lesssim s^{-1}\left[\rho\Big(\bigcup_{i \in I_\alpha} (R_i^\alpha)^*\Big) + \rho\Big(\Big\{x \notin \bigcup_{i \in I_\alpha} (R_i^\alpha)^* : T^*b^\alpha(x) > (C_{\kappa_0}C_3\nu_1 + 3)\alpha\Big\}\Big)\right] \\
&\lesssim s^{-1} \sum_{i \in I_\alpha} \rho\left(R_i^\alpha\right) \lesssim s^{-1}\alpha^{-p}\|f\|^p_{L^p}.
\end{aligned}$$

The estimates of I and II above imply that for any given $s \in (0,1)$,

$$\|\mathcal{M}^\sharp_{0,s}(T^*f)\|_{L^{p,\infty}} \lesssim s^{-1/p}\|f\|_{L^p}. \tag{4.11}$$

By the assumption $f \in L^\infty_{c,0}$ and Lemma 4.2, $T^*f \in L^{1,\infty}$. Then by applying Proposition 3.2 to the function T^*f and by (4.11), we obtain that for $s \in (0, 1/2]$ such that $s < (2^2 3^p C_2)^{-1}$,

$$\|T^*f\|_{L^{p,\infty}} \le C\|\mathcal{M}^\sharp_{0,s}(T^*f)\|_{L^{p,\infty}} \le C\|f\|_{L^p}.$$

This proves (4.10). Thus, (ii) holds.

It is obvious that (ii) implies (iii). Now we assume that (iii) holds for an index $p \in (1,\infty)$ and show that (iv) holds. To this end, for any given $f \in L^1$ and $\alpha > 0$, we use Proposition 2.2 to obtain a sequence of mutually disjoint sets, $\{R_i^\alpha\}_{i \in I_\alpha} \subset \mathcal{R}$, and a decomposition of f as $f = g^\alpha + b^\alpha = g^\alpha + \sum_{i \in I_\alpha} b_i^\alpha$ where $\|g^\alpha\|_{L^\infty} \lesssim \alpha$, every b_i^α is supported on R_i^α and has integral 0, $\frac{1}{\rho(R_i^\alpha)}\int_{R_i^\alpha} |f|\,d\rho \approx \alpha$ and $\|b_i^\alpha\|_{L^1} \lesssim \alpha\rho(R_i^\alpha)$. By Lemma 4.1 and Lemma 2.1(ii), there exists a sufficiently large positive constant C such that for all $\alpha > 0$,

$$\rho\big(\{x \in S : T^*f(x) > (C+1)\alpha\}\big)$$

$$
\begin{aligned}
&\le \rho\big(\{x \in S : T^* g^\alpha(x) > \alpha\}\big) + \rho\left(\bigcup_{i\in I_\alpha} (R_i^\alpha)^*\right) \\
&\quad + \rho\left(\left\{x \notin \bigcup_{i\in I_\alpha} (R_i^\alpha)^* : T^* b^\alpha(x) > C\alpha\right\}\right) \\
&\lesssim \alpha^{-p} \|T^* g^\alpha\|_{L^p}^p + \sum_{i\in I_\alpha} \rho(R_i^\alpha).
\end{aligned}
$$

Notice that $\sum_{i\in I_\alpha} \rho(R_i^\alpha) \lesssim \alpha^{-1}\|f\|_{L^1}$. Using the L^p-boundedness of T^* (by (iii)) and the properties of g^α, we have

$$
\alpha^{-p}\|T^* g^\alpha\|_{L^p}^p \lesssim \alpha^{-p}\|T^*\|_{L^p\to L^p}^p \|g^\alpha\|_{L^p}^p \lesssim \alpha^{-1}\|g^\alpha\|_{L^1} \lesssim \alpha^{-1}\|f\|_{L^1}.
$$

Combining all the above estimates yields that $\|T^* f\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}$. Hence, (iv) holds.

Finally, we show that (iv) implies (i). Fix $\sigma \in (0,1)$. By Lemma 3.3, it suffices to prove that for all $f \in L_c^\infty$, $\|T^* f\|_{*,\sigma} \lesssim \|f\|_{L^\infty}$. To this end, for any given $f \in L_c^\infty$ and $R \in \mathcal{R}$, we decompose f into $f = f\chi_{R^*} + f\chi_{S\setminus R^*} \equiv f_1 + f_2$. Notice that for all $c \in \mathbb{C}$ and $x \in S$, $|T^* f(x) - c| \le T^* f_1(x) + |T^* f_2(x) - c|$. Then, for all $R \in \mathcal{R}(x)$,

$$
\begin{aligned}
&\inf_{c\in\mathbb{C}} \frac{1}{\rho(R)} \int_R |T^* f(x) - c|^\sigma \, d\rho(x) \\
&\quad \le \frac{1}{\rho(R)} \int_R |T^* f_1(x)|^\sigma \, d\rho(x) + \inf_{c\in\mathbb{C}} \frac{1}{\rho(R)} \int_R |T^* f_2(x) - c|^\sigma \, d\rho(x) \equiv \mathrm{Z}_1 + \mathrm{Z}_2. \quad (4.12)
\end{aligned}
$$

Using $\sigma \in (0,1)$ and the hypothesis that T^* is bounded from L^1 to $L^{1,\infty}$ together with Lemma 2.1(ii), we obtain

$$
\begin{aligned}
\mathrm{Z}_1 &= \frac{1}{\rho(R)} \int_0^\infty \sigma t^{\sigma-1} \rho\left(\{x \in R : T^* f_1(x) > t\}\right) dt \\
&\le \frac{1}{\rho(R)} \left\{ \int_0^{\|f\|_{L^\infty}} \sigma t^{\sigma-1} \rho(R)\, dt + \int_{\|f\|_{L^\infty}}^\infty \sigma t^{\sigma-1} \frac{\|T^*\|_{L^1\to L^{1,\infty}} \|f_1\|_{L^1}}{t} \, dt \right\} \lesssim \|f\|_{L^\infty}^\sigma.
\end{aligned}
$$

By this and (4.12), the proof of (i) is reduced to the estimate $\mathrm{Z}_2 \lesssim \|f\|_{L^\infty}^\sigma$. Since $f_2 \in L_c^\infty$, an argument similar to the one used for Z_1 yields $|T^* f_2|^\sigma \in L^1_{\mathrm{loc}}$; thus there exists some $z_R \in R$ such that $T^* f_2(z_R) < \infty$. Notice that for all $x \in R$ and $\widetilde{R} \in \mathcal{R}(x)$, Lemma 2.1(i) implies that $\{y \in S : d(y,x) > r_{\widetilde{R}}\} = (\widetilde{R}^*)^\complement \cup \{y \in \widetilde{R}^* : d(y,x) > r_{\widetilde{R}}\}$. From this, it follows that for all $x \in R$, we can write $T^* f_2(x)$ as follows:

$$
\begin{aligned}
T^* f_2(x) = \sup_{\widetilde{R}\in\mathcal{R}(x)} \Bigg| &\int_{d(y,x)>r_{\widetilde{R}}} K(x,y) f_2(y)\, d\rho(y) \\
&- \int_{\substack{d(y,x)>r_{\widetilde{R}} \\ y\in\widetilde{R}^*}} K(x,y) f_2(y)\, d\rho(y) \Bigg|. \quad (4.13)
\end{aligned}
$$

In particular, the equality (4.13) holds for $T^*f_2(z_R)$. Thus, for all $x \in R$,

$$\begin{aligned}
&|T^*f_2(x) - T^*f_2(z_R)| \\
&\quad \le \left| \sup_{\widetilde{R}\in\mathcal{R}(x)} \left| \int_{d(y,x)>r_{\widetilde{R}}} K(x,y) f_2(y)\, d\rho(y) \right| - \sup_{\widetilde{R}\in\mathcal{R}(z_R)} \left| \int_{d(y,z_R)>r_{\widetilde{R}}} K(z_R,y) f_2(y)\, d\rho(y) \right| \right| \\
&\qquad + \sup_{\widetilde{R}\in\mathcal{R}(x)} \left| \int_{\substack{d(y,x)>r_{\widetilde{R}} \\ y\in\widetilde{R}^*}} K(x,y) f_2(y)\, d\rho(y) \right| + \sup_{\widetilde{R}\in\mathcal{R}(z_R)} \left| \int_{\substack{d(y,z_R)>r_{\widetilde{R}} \\ y\in\widetilde{R}^*}} K(z_R,y) f_2(y)\, d\rho(y) \right| \\
&\quad \equiv \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3.
\end{aligned}$$

To see this, by symmetry, it suffices to show that $T^*f_2(x) - T^*f_2(z_R) \le L_1 + L_2 + L_3$, which follows by first writing $T^*f_2(x)$ and $T^*f_2(z_R)$ as in (4.13), then applying

$$\sup_{i\in\Lambda} |a_i - b_i| \le \sup_{i\in\Lambda} |a_i| + \sup_{i\in\Lambda} |b_i|$$

to the expression of $T^*f_2(x)$, and $\sup_{i\in\Lambda} |a_i - b_i| \ge \sup_{i\in\Lambda} |a_i| - \sup_{i\in\Lambda} |b_i|$ in the expression of $T^*f_2(z_R)$, where Λ denotes an index set which might be uncountable.

When $d(y,x) > r_{\widetilde{R}}$ and $y \in \widetilde{R}^*$, by $x \in R \cap \widetilde{R}$ and Lemma 2.1(i), we have $r_{\widetilde{R}} < d(y,x) \le (2\kappa_0 + 1) r_{\widetilde{R}}$, which combined with (4.1) implies that $\mathrm{L}_2 \lesssim \nu_1 \|f\|_{L^\infty}$. Similarly, $\mathrm{L}_3 \lesssim \nu_1 \|f\|_{L^\infty}$. To estimate L_1, by the properties of Calderón–Zygmund sets, we obtain

$$\sup_{\widetilde{R}\in\mathcal{R}(x)} \left| \int_{d(y,x)>r_{\widetilde{R}}} K(x,y) f_2(y)\, d\rho(y) \right| = \sup_{\epsilon>0} \left| \int_{d(y,x)>\epsilon} K(x,y) f_2(y)\, d\rho(y) \right|.$$

By this and the inequality $\sup_{i\in\Lambda} |a_i - b_i| \ge \left| \sup_{i\in\Lambda} |a_i| - \sup_{i\in\Lambda} |b_i| \right|$, we obtain that

$$\begin{aligned}
\mathrm{L}_1 &\le \sup_{\epsilon>0} \left| \int_{d(y,x)>\epsilon} K(x,y) f_2(y)\, d\rho(y) - \int_{d(y,z_R)>\epsilon} K(z_R,y) f_2(y)\, d\rho(y) \right| \\
&\le \sup_{\epsilon>0} \int_{d(x,y)>\epsilon,\, d(z_R,y)>\epsilon} |K(x,y) - K(z_R,y)| |f_2(y)|\, d\rho(y) \\
&\quad + \sup_{\epsilon>0} \int_{d(x,y)>\epsilon\ge d(z_R,y)} |K(x,y) f_2(y)|\, d\rho(y) \\
&\quad + \sup_{\epsilon>0} \int_{d(z_R,y)>\epsilon\ge d(x,y)} |K(z_R,y) f_2(y)|\, d\rho(y) \equiv \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3.
\end{aligned}$$

From (4.2) and $\operatorname{supp} f_2 \subset (R^*)^{\complement}$, it follows that $\mathrm{J}_1 \lesssim \nu_2 \|f\|_{L^\infty}$. If $x \in R$, $y \notin R^*$ and $d(x,y) > \epsilon \ge d(z_R, y)$, by Lemma 2.1(i), we have $r_R < d(y,R) \le d(y,z_R) \le \epsilon$ and

$$d(x,y) \le d(x,z_R) + d(z_R,y) < 2\kappa_0 r_R + \epsilon < (2\kappa_0 + 1)\epsilon,$$

which together with (4.1) implies that $\mathrm{J}_2 \lesssim \nu_1 \|f\|_{L^\infty}$. Similarly, $\mathrm{J}_3 \lesssim \nu_1 \|f\|_{L^\infty}$. Thus, $\mathrm{L}_1 = \sum_{i=1}^3 \mathrm{J}_i \lesssim \|f\|_{L^\infty}$. Combining the estimates of $\mathrm{L}_1, \mathrm{L}_2$ and L_3 yields $\mathrm{Z}_2 \lesssim \|f\|_{L^\infty}^\sigma$. This finishes the proof of (iv) implies (i), and hence the proof of Theorem 4.1. $\square$

Applying Theorem 4.1 and the Calderón–Zygmund decomposition, we obtain the following result.

Theorem 4.2. *Let T be the integral operator associated with a kernel K satisfying* (4.1) *and* (4.2)*. If T is bounded on L^2, then the maximal singular integral T^* defined as in* (4.3) *is bounded from L^1 to $L^{1,\infty}$, from L^p to L^p for all $p \in (1, \infty)$, and from L_c^∞ to* BMO .

To prove Theorem 4.2, we need the following Cotlar-type inequality.

Lemma 4.3. *Under the assumptions of Theorem 4.2, for all $g \in L_c^\infty$ and $x \in S$,*

$$T^*g(x) \le \mathcal{M}(Tg)(x) + \left[\nu_2 + \kappa_0^{1/2}\|T\|_{L^2\to L^2}\right]\|g\|_{L^\infty}, \tag{4.14}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator defined in (2.1)*.*

Proof. We only give an outline of the proof because of its similarity to the argument used in [12, Theorem 1]; see also [13, p. 295]. Indeed, for all $x \in S$, $R \in \mathcal{R}(x)$, $z \in R$ and all $g \in L_c^\infty$, we use Hörmander's condition (4.2) to obtain

$$\begin{aligned}|T_R g(x)| &\le |T(g\chi_{(R^*)^\complement})(x) - T(g\chi_{(R^*)^\complement})(z)| + |Tg(z)| + |T(g\chi_{R^*})(z)|\\ &\le \nu_2\|g\|_{L^\infty} + |Tg(z)| + |T(g\chi_{R^*})(z)|.\end{aligned}$$

Taking the integral average over R with respect to the variable z in both sides of this inequality yields that

$$|T_R g(x)| \le \nu_2\|g\|_{L^\infty} + \frac{1}{\rho(R)}\int_R |Tg(z)|\,d\rho(z) + \frac{1}{\rho(R)}\int_R |T(g\chi_{R^*})(z)|\,d\rho(z).$$

By (2.1), Hölder's inequality, the L^2-boundedness of T and Lemma 2.1(ii), we obtain

$$\begin{aligned}|T_R g(x)| &\le \nu_2\|g\|_{L^\infty} + \mathcal{M}(Tg)(x) + \left\{\frac{1}{\rho(R)}\int_R |T(g\chi_{R^*})(z)|^2\,d\rho(z)\right\}^{1/2}\\ &\le \mathcal{M}(Tg)(x) + \left[\nu_2 + \kappa_0^{1/2}\|T\|_{L^2\to L^2}\right]\|g\|_{L^\infty},\end{aligned}$$

which completes the proof. □

Proof of Theorem 4.2. By Theorem 4.1, it suffices to show that T^* is bounded from L^1 to $L^{1,\infty}$. To this end, for any $f \in L^1$ with bounded support and $\alpha > 0$, we decompose f at level α into $f = g^\alpha + b^\alpha = g^\alpha + \sum_{i\in I_\alpha} b_i^\alpha$, where I_α is a certain index set, $\|g^\alpha\|_{L^\infty} \le C_1\alpha$, $\operatorname{supp} b_i^\alpha \subset R_i^\alpha$, $\{R_i^\alpha\}_{i\in I_\alpha} \subset \mathcal{R}$ are mutually disjoint, $\int_S b_i^\alpha\,d\rho = 0$, $\alpha \le \frac{1}{\rho(R_i^\alpha)}\int_{R_i^\alpha}|f|\,d\rho \le C_1\alpha$ and $\|b_i^\alpha\|_{L^1} \le C_1\alpha\rho(R_i^\alpha)$. Here C_1 is the constant which appears in Proposition 2.2. For $C_0 > C_{\kappa_0}C_1\nu_1 + 3$ sufficiently large, which will be determined later, we have

$$\begin{aligned}\rho(\{x \in S : T^*f(x) > C_0\alpha\}) &\le \rho(\{x \in S : T^*g^\alpha(x) > (C_0 - C_{\kappa_0}C_1\nu_1 - 3)\alpha\})\\ &\quad + \rho(\{x \in S : T^*b^\alpha(x) > (C_{\kappa_0}C_1\nu_1 + 3)\alpha\}) \equiv \mathrm{Z}_1 + \mathrm{Z}_2.\end{aligned}$$

Here C_{κ_0} is as in Lemma 4.1. To estimate Z_1, the inequality (4.14) applied to to g^α gives that

$$T^* g^\alpha(x) \le \mathcal{M}(Tg^\alpha)(x) + C_1 \left[\nu_2 + \kappa_0^{1/2} \|T\|_{L^2 \to L^2}\right] \alpha.$$

Set $\widetilde{C} \equiv C_1[\nu_2 + \kappa_0^{1/2}\|T\|_{L^2\to L^2}] + C_{\kappa_0} C_1 \nu_1 + 3$. If $C_0 > \widetilde{C}$, by the L^2-boundedness of $\mathcal{M}$ and T, the facts that $\|g^\alpha\|_{L^\infty} \lesssim \alpha$ and $\|g^\alpha\|_{L^1} \lesssim \|f\|_{L^1}$, we obtain

$$\mathrm{Z}_1 \lesssim \alpha^{-2} \|\mathcal{M}(Tg^\alpha)\|_{L^2}^2 \lesssim \alpha^{-2} \|g^\alpha\|_{L^2}^2 \lesssim \alpha^{-1} \|f\|_{L^1}.$$

Lemma 4.1 implies that $\mathrm{Z}_2 \lesssim \sum_{i \in I_\alpha} \rho(R_i^\alpha) \lesssim \alpha^{-1}\|f\|_{L^1}$. By the estimates of Z_1 and Z_2, we have that T^* maps all L^1 functions with bounded support into $L^{1,\infty}$. A standard density argument implies the boundedness of T^* from L^1 to $L^{1,\infty}$. This concludes the proof. $\square$

5 Applications to Multiplier Operators on $ax + b$−Groups

The aim of this section is to apply the results in Section 4 to the multipliers of a distinguished Laplacian Δ on (S, d, ρ). Let us begin with some known facts related to the integration formulae and spherical analysis on S; for details we refer the reader to [1, 2, 6, 7, 17].

A *radial function* on S is a function that depends only on the distance from the identity. If f is radial and $f \in C_c^\infty(S)$, then we have the following integration formula:

$$\int_S f(x)\, d\lambda(x) = C \int_0^\infty f(r) A(r)\, dr, \tag{5.1}$$

where C is a positive constant depending only on S, $A(r) = 4^n \sinh^n\left(\frac{r}{2}\right) \cosh^n\left(\frac{r}{2}\right)$ for all $r > 0$ and λ denotes the *left Haar measure*. One easily checks that

$$A(r) \lesssim \left(\frac{r}{1+r}\right)^n e^{nr} \qquad \forall\, r > 0. \tag{5.2}$$

A radial function ϕ is *spherical* if it is an eigenfunction of the *Laplace-Beltrami operator* $\mathcal{L} \equiv -\mathrm{div} \cdot \mathrm{grad}$ and $\phi(e) = 1$. For $s \in \mathbb{C}$, let ϕ_s be the *spherical function with eigenvalue* $s^2 + n^2/4$. It is known ([2]) that the spherical function ϕ_0 satisfies the estimate

$$0 < \phi_0(r) \lesssim (1 + r) e^{-nr/2} \qquad \forall\, r > 0, \tag{5.3}$$

and that for every radial function $f \in C_c^\infty(S)$,

$$\int_S \delta^{1/2}(x) f(x)\, d\rho(x) = \int_0^\infty \phi_0(r) f(r) A(r)\, dr. \tag{5.4}$$

The *spherical Fourier transform* of a radial function f in $L^1(\lambda)$ is defined by the formula

$$\mathcal{H}f(s) \equiv \int_S \phi_s(x) f(x)\, d\lambda(x) \qquad \forall\, s \in \mathbb{C}.$$

For radial functions $f \in C_c^\infty(S)$, a *Plancherel formula* holds:

$$\int_S |f(x)|^2\, d\lambda(x) = C \int_0^\infty |\mathcal{H}f(s)|^2 |\mathbf{c}(s)|^{-2}\, ds, \tag{5.5}$$

where C is a positive constant depending only on S, and $|\mathbf{c}(s)|^{-2}\,ds$ denotes the *Plancherel measure* which satisfies the following estimates (see Chapter IV of [17]):

$$|\mathbf{c}(s)|^{-2} \leq \begin{cases} C|s|^2 & \text{if } |s| \leq 1, \\ C|s|^n & \text{if } |s| > 1, \end{cases} \tag{5.6}$$

where C is a positive constant independent of s. In particular, when $n=1$, the estimate (5.6) becomes

$$|\mathbf{c}(s)|^{-2} \leq C \min\{|s|^2,\, |s|\} \qquad \forall\, s \in \mathbb{R}^+\,, \tag{5.7}$$

where C is a positive constant independent of s.

Denote by $\mathcal{A}$ the *Abel transform* and by $\mathcal{A}^{-1}$ the *inverse Abel transform*. If n is even, then

$$\mathcal{A}^{-1}f(r) = (2\pi)^{-n/2}\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n/2} f(r) \qquad \forall\, r > 0, \tag{5.8}$$

and if n is odd, then for all $r>0$,

$$\mathcal{A}^{-1}f(r) = (2\pi)^{-n/2}\int_r^\infty \left[\left(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{(n+1)/2} f(s)\right](\cosh s - \cosh r)^{-1/2}\sinh s\, ds. \tag{5.9}$$

Denote by $\mathcal{F}(g)$ or $\widehat{g}$ the *Fourier transform* of g on $\mathbb{R}$, namely, $\mathcal{F}g(s) = \int_{\mathbb{R}} g(r)e^{-isr}\,dr$. It is known that $\mathcal{H} = \mathcal{F}\circ\mathcal{A}$, and hence $\mathcal{H}^{-1} = \mathcal{A}^{-1}\circ\mathcal{F}^{-1}$.

Consider the following *basis of left-invariant vector fields of the Lie algebra of* S:

$$X_0 \equiv a\partial_a, \quad X_i \equiv a\partial_{x_i}, \quad i = 1,\, 2,\, \cdots,\, n.$$

The *Laplacian* $\Delta \equiv -\sum_{i=0}^n X_i^2$ is a left-invariant essentially selfadjoint operator on $L^2(\rho)$. The operator Δ has a special relationship with the Laplace–Beltrami operator $\mathcal{L}$ associated with the Riemannian structure of S. Indeed, if we denote by $\mathcal{L}_n$ the *shifted operator* $\mathcal{L} - n^2/4$, it is known that

$$\delta^{-1/2} \triangle\, \delta^{1/2} f = \mathcal{L}_n f \tag{5.10}$$

for all smooth radial functions f on S (see [2]), where δ denotes the *modular function*. The *spectra* of both Δ on $L^2(\rho)$ and $\mathcal{L}_n$ on $L^2(\lambda)$ are $[0,\infty)$. Let E_Δ and $E_{\mathcal{L}_n}$ be the *spectral resolutions of identity* for which $\triangle = \int_0^\infty t\, dE_\triangle(t)$ and $\mathcal{L}_n = \int_0^\infty t\, dE_{\mathcal{L}_n}(t)$. For each bounded measurable function m on $\mathbb{R}^+$, the *operators* $m(\triangle)$ and $m(\mathcal{L}_n)$, spectrally defined by

$$m(\triangle) = \int_0^\infty m(t)\, dE_\triangle(t) \quad \text{and} \quad m(\mathcal{L}_n) = \int_0^\infty m(t)\, dE_{\mathcal{L}_n}(t),$$

are respectively bounded on $L^2(\rho)$ and $L^2(\lambda)$ by the spectral theorem. By (5.10) and the spectral theorem, we see that for all radial functions $f \in C_c^\infty(S)$,

$$\delta^{-1/2} m(\triangle)\delta^{1/2} f = m(\mathcal{L}_n) f.$$

Denote by $k_{m(\triangle)}$ the *convolution kernel* of $m(\triangle)$, namely, for all $f \in C_c^\infty(S)$,

$$m(\triangle)f(x) = \int_S f(xy^{-1})k_{m(\triangle)}(y)\,d\rho(y) \qquad \forall\, x \notin \operatorname{supp} f. \tag{5.11}$$

As in (5.11), denote by $k_{m(\mathcal{L}_n)}$ the *convolution kernel* of $m(\mathcal{L}_n)$. It was proved in [1, 2] that for all bounded measurable function m on $\mathbb{R}^+$, the convolution kernel $k_{m(\mathcal{L}_n)}$ is radial,

$$k_{m(\triangle)} = \delta^{1/2}k_{m(\mathcal{L}_n)} \text{ and } \mathcal{H}k_{m(\mathcal{L}_n)}(s) = m(s^2) \qquad \forall\, s \in \mathbb{R}^+. \tag{5.12}$$

Let $K_{m(\triangle)}$ be the *integral kernel* of $m(\triangle)$, namely, for all $f \in C_c^\infty(S)$,

$$m(\triangle)f(x) = \int_S K_{m(\triangle)}(x,y)f(y)\,d\rho(y) \qquad \forall\, x \notin \operatorname{supp} f. \tag{5.13}$$

In view of (5.11) and (5.13), by changing variables and using the left-invariant property of λ and the right-invariant property of ρ, we obtain that for all x, $y \in S$,

$$K_{m(\triangle)}(x,y) = k_{m(\triangle)}(y^{-1}x)\delta(y). \tag{5.14}$$

For any $s \in (0,\infty)$, we denote by $H^s(\mathbb{R})$ the *Sobolev space* $W^{s,2}(\mathbb{R})$ of order s on $\mathbb{R}$. Let $\phi \in C_c^\infty(\mathbb{R}^+)$ be a function supported in $[1/4, 4]$ such that

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}t) = 1 \quad \forall\, t \in \mathbb{R}^+. \tag{5.15}$$

For any given s_0, $s_\infty \in \mathbb{R}^+$, a bounded measurable function m on $\mathbb{R}^+$ is said to satisfy a *mixed Mihlin-Hörmander condition of order* (s_0, s_∞) if

$$\|m\|_{s_0} \equiv \sup_{t<1} \|m(t\cdot)\phi(\cdot)\|_{H^{s_0}(\mathbb{R})} < \infty \text{ and } \|m\|_{s_\infty} \equiv \sup_{t\geq 1} \|m(t\cdot)\phi(\cdot)\|_{H^{s_\infty}(\mathbb{R})} < \infty.$$

For any $j \in \mathbb{Z}$ and any bounded measurable function m on $\mathbb{R}^+$, we define m_j by

$$m_j(t) \equiv m(2^j t)\phi(t) \qquad \forall\, t \in \mathbb{R}^+. \tag{5.16}$$

Obviously, we have

$$m(\triangle) = \sum_{j\in\mathbb{Z}} m_j(2^{-j}\triangle). \tag{5.17}$$

As in (5.11) and (5.13), we denote by $k_{m_j(2^{-j}\triangle)}$ and $K_{m_j(2^{-j}\triangle)}$ the *convolution kernel* and the *integral kernel* of $m_j(2^{-j}\triangle)$, respectively.

Assume that m satisfies a mixed Mihlin-Hörmander condition of order (s_0, s_∞) with $s_0 > 3/2$ and $s_\infty > \max\{3/2, (n+1)/2\}$. Choose $\sigma > 0$ small enough such that $s_0 > 3/2+\sigma$ and $s_\infty > \max\{3/2, (n+1)/2\} + \sigma$. Hebisch and Steger [16, Theorems 2.4 and 6.1] proved that there exists a positive constant C such that for all $j \in \mathbb{Z}$ and $y \in S$,

$$\int_S |K_{m_j(2^{-j}\triangle)}(x,y)|(1 + 2^{j/2}d(x,y))^\sigma\,d\rho(x) \leq \begin{cases} C\|m\|_{s_0} & \text{if } j \leq 0, \\ C\|m\|_{s_\infty} & \text{if } j > 0; \end{cases} \tag{5.18}$$

and that for all y, $z \in S$,

$$\begin{aligned}&\int_S |K_{m_j(2^{-j}\triangle)}(x,y) - K_{m_j(2^{-j}\triangle)}(x,z)|\,d\rho(x)\\&\quad\le \begin{cases} C2^{j/2}d(y,z)\|m\|_{s_0} & \text{if } j \le 0,\\ C2^{j/2}d(y,z)\|m\|_{s_\infty} & \text{if } j > 0.\end{cases}\end{aligned}\tag{5.19}$$

From (5.18) and (5.19), it is easy to deduce that

$$\sup_{R\in\mathcal{R}}\sup_{y,\,z\in R}\int_{S\setminus R^*} |K_{m(\triangle)}(x,y) - K_{m(\triangle)}(x,z)|\,d\rho(x) < \infty;\tag{5.20}$$

see [16, Remark 1.4]. From the proofs of [16, Theorems 2.4 and 6.1], it follows that (5.18) and (5.19) still hold if we interchange the two variables of $K_{m_j(2^{-j}\triangle)}$. Thus, $K_{m(\triangle)}$ satisfies Hörmander's condition (4.2).

The estimates (5.18) and (5.19) imply that the operator $m(\triangle)$ is bounded from L^1 to $L^{1,\infty}$ and bounded on L^p for all $p \in (1,\infty)$ [16, Theorem 2.4] and that it is also bounded from H^1 to L^1 and from L^∞ to BMO [32, Proposition 2.4].

We now consider the boundedness of the maximal singular integral operator $(m(\triangle))^*$ as defined in (4.3).

Theorem 5.1. *Let m satisfy a mixed Mihlin-Hörmander condition of order (s_0, s_∞) with $s_0 > 3/2$ and $s_\infty > \max\{3/2, (n+1)/2\}$. Then the maximal singular integral $(m(\triangle))^*$ is bounded from L^1 to $L^{1,\infty}$, from L^∞_c to* BMO *and bounded on L^p for all $p \in (1,\infty)$.*

Theorem 5.1 follows immediately from Theorem 4.2 if we know that $K_{m(\triangle)}$ satisfies (4.1) and (4.2). We already noticed that the kernel $K_{m(\triangle)}$ satisfies Hörmander's condition (4.2). Condition (4.1) for $K_{m(\triangle)}$ is equivalent to the following estimate:

$$\sup_{\epsilon>0}\int_{\epsilon<d(y,e)\le 2\epsilon} |k_{m(\triangle)}(y)|\,d\rho(y) < \infty.\tag{5.21}$$

To see this, by using the right-invariant property of ρ and the left-invariant property of λ, we have that for all $x, y \in S$,

$$\delta(y)d\rho(y) = d\lambda(y) = d\rho(y^{-1}) = d\rho(y^{-1}x) \quad\text{and}\quad d(x,y) = d(y^{-1}x, e).$$

We then apply (5.14) to obtain

$$\begin{aligned}\int_{\epsilon<d(x,y)\le 2\epsilon} |K_{m(\triangle)}(x,y)|\,d\rho(y) &= \int_{\epsilon<d(x,y)\le 2\epsilon} |k_{m(\triangle)}(y^{-1}x)|\delta(y)\,d\rho(y)\\ &= \int_{\epsilon<d(y^{-1}x,e)\le 2\epsilon} |k_{m(\triangle)}(y^{-1}x)|\,d\rho(y^{-1}x)\\ &= \int_{\epsilon<d(y,e)\le 2\epsilon} |k_{m(\triangle)}(y)|\,d\rho(y).\end{aligned}$$

Similarly, by (5.14) and

$$\delta(y)d\rho(x) = \delta(y)[\delta(x)]^{-1}d\lambda(x) = [\delta(y^{-1}x)]^{-1}d\lambda(x) = [\delta(y^{-1}x)]^{-1}d\lambda(y^{-1}x) = d\rho(y^{-1}x),$$

we have

$$\begin{aligned}\int_{\epsilon<d(x,y)\le 2\epsilon} |K_{m(\triangle)}(x,y)|\,d\rho(x) &= \int_{\epsilon<d(x,y)\le 2\epsilon} |k_{m(\triangle)}(y^{-1}x)|\delta(y)\,d\rho(x)\\ &= \int_{\epsilon<d(y^{-1}x,e)\le 2\epsilon} |k_{m(\triangle)}(y^{-1}x)|\,d\rho(y^{-1}x)\\ &= \int_{\epsilon<d(y,e)\le 2\epsilon} |k_{m(\triangle)}(y)|\,d\rho(y).\end{aligned}$$

Therefore, it suffices to show (5.21). To this end, we will use some ideas from the proof of [34, Theorem 4.3]. Let us start with an integral estimate of the kernel which is more delicate than the one proved in [34, Lemma 4.5].

Lemma 5.1. *Let f be a bounded even function on $\mathbb{R}$ such that $\operatorname{supp}\widehat{f}\subset[-r,r]$. Then, there exists a positive constant C independent of r such that $k_{f(\sqrt{\triangle})}$ satisfies the following:*

(i) *if $\epsilon\ge r$, then $\int_{\epsilon<d(x,e)\le 2\epsilon}|k_{f(\sqrt{\triangle})}(x)|\,d\rho(x)=0$;*

(ii) *for all $\epsilon\in(0,1)$ such that $\epsilon<r$,*

$$\int_{\epsilon<d(x,e)\le 2\epsilon} |k_{f(\sqrt{\triangle})}(x)|\,d\rho(x) \le C\epsilon^{(n+1)/2}\left\{\int_0^\infty |f(s)|^2(s^2+s^n)\,ds\right\}^{1/2}; \qquad (5.22)$$

(iii) *for all $\epsilon\in[1,\infty)$ such that $\epsilon<r$,*

$$\int_{\epsilon<d(x,e)\le 2\epsilon} |k_{f(\sqrt{\triangle})}(x)|\,d\rho(x) \le C\epsilon^{3/2}\left\{\int_0^\infty |f(s)|^2(s^2+s^n)\,ds\right\}^{1/2}; \qquad (5.23)$$

(iv) *when $n=1$, the right-hand sides of (5.22) and (5.23) can be respectively replaced by the better estimates: $C\epsilon^{(n+1)/2}\left\{\int_0^\infty |f(s)|^2\min\{s,s^2\}\,ds\right\}^{1/2}$ and*

$$C\epsilon^{3/2}\left[\int_0^\infty |f(s)|^2\min\{s,s^2\}\,ds\right]^{1/2}.$$

Proof. Let $k_{f(\sqrt{\mathcal{L}_n})}$ be the *convolution kernel* of the operator $f(\sqrt{\mathcal{L}_n})$. By (5.12), we know that $k_{f(\sqrt{\mathcal{L}_n})}$ is radial on S, $k_{f(\sqrt{\triangle})}=\delta^{1/2}k_{f(\sqrt{\mathcal{L}_n})}$ and $\mathcal{H}k_{f(\sqrt{\mathcal{L}_n})}(t)=f(t)$ for all $t\in\mathbb{R}^+$. Since $\mathcal{H}^{-1}=\mathcal{A}^{-1}\circ\mathcal{F}^{-1}$ and f is even, we obtain

$$k_{f(\sqrt{\mathcal{L}_n})}(t)=\mathcal{H}^{-1}f(t)=\mathcal{A}^{-1}\mathcal{F}^{-1}f(t)=\mathcal{A}^{-1}(\widehat{f}\,)(t). \qquad (5.24)$$

Notice that $\operatorname{supp}\widehat{f}\subset[-r,r]$ and the inverse formulae for the Abel transform (5.8) and (5.9) imply that $\operatorname{supp}\mathcal{A}^{-1}(\widehat{f}\,)\subset B(e,r)$. Then, $\operatorname{supp}k_{f(\sqrt{\mathcal{L}_n})}\subset B(e,r)$, and hence

$\operatorname{supp} k_{f(\sqrt{\triangle})} \subset B(e,r)$. So the integral of $k_{f(\sqrt{\triangle})}$ in the domain $\{x \in S : \epsilon < d(x,e) \le 2\epsilon\}$ is 0 when $\epsilon \ge r$. This proves (i).

To prove (ii) and (iii), we set $w(x) \equiv \delta^{-1/2}(x)e^{nd(x,e)/2}$ for all $x \in S$. Applying Hölder's inequality yields that

$$
\begin{aligned}
&\int_{\epsilon<d(x,e)\le 2\epsilon} |k_{f(\sqrt{\triangle})}(x)|\, d\rho(x) \\
&\le \left\{ \int_{\epsilon<d(x,e)\le 2\epsilon} w(x)^{-1}\, d\rho(x) \right\}^{1/2} \left\{ \int_{\epsilon<d(x,e)\le 2\epsilon} |k_{f(\sqrt{\triangle})}(x)|^2\, w(x)\, d\rho(x) \right\}^{1/2} \\
&\equiv \mathrm{I}^{1/2} \cdot \mathrm{J}^{1/2},
\end{aligned}
$$

where we denoted by I and J respectively the integral in the first and second bracket. Recall that if $x = (y,a) \in S$ with $y \in \mathbb{R}^n$ and $a \in \mathbb{R}^+$, then $\delta(x) = \delta(y,a) = a^{-n}$. When $\epsilon \in (0,1]$, for all $d(x,e) \le 2\epsilon$, we have $|\delta(x)| \lesssim \epsilon \lesssim 1$, and hence

$$
\mathrm{I} = \int_{\epsilon<d(x,e)\le 2\epsilon} \delta^{1/2}(x) e^{-nd(x,e)/2}\, d\rho(x) \lesssim \rho\big(B(e,2\epsilon)\big) \lesssim \epsilon^{n+1}.
$$

When $\epsilon > 1$, by (5.4) together with the estimates (5.3) and (5.2) of ϕ_0 and of the density function A, we obtain

$$
\begin{aligned}
\mathrm{I} &\le \int_{d(x,e)\le 2\epsilon} \delta^{1/2}(x) e^{-nd(x,e)/2} d\rho(x) \\
&= \int_0^{2\epsilon} \phi_0(t) e^{-nt/2} A(t)\, dt \lesssim \int_0^{2\epsilon} (1+t) \left(\frac{t}{1+t} \right)^n dt \lesssim \epsilon^2.
\end{aligned}
$$

To estimate J, since $k_{f(\sqrt{\triangle})} = \delta^{1/2} k_{f(\sqrt{\mathcal{L}_n})}$, we have

$$
\mathrm{J} = \int_{\epsilon<d(x,e)\le 2\epsilon} \delta(x) |k_{f(\sqrt{\mathcal{L}_n})}(x)|^2\, \delta^{-1/2}(x) e^{nd(x,e)/2}\, d\rho(x).
$$

Again, using (5.4) and the estimates (5.3) and (5.2), we estimate J by

$$
\begin{aligned}
\mathrm{J} &= \int_\epsilon^{2\epsilon} \phi_0(t) |k_{f(\sqrt{\mathcal{L}_n})}(t)|^2 e^{nt/2} A(t)\, dt \\
&\lesssim (1+\epsilon) \int_\epsilon^{2\epsilon} |k_{f(\sqrt{\mathcal{L}_n})}(t)|^2 A(t)\, dt \lesssim (1+\epsilon) \int_S |k_{f(\sqrt{\mathcal{L}_n})}(x)|^2\, d\lambda(x),
\end{aligned}
$$

where the last inequality is due to (5.1). Applying the Plancherel formula (5.5) and the estimate for the Plancherel measure (5.6) (when $n = 1$ we use (5.7) instead) yields that

$$
\begin{aligned}
\int_S |k_{f(\sqrt{\mathcal{L}_n})}(x)|^2\, d\lambda(x) &\approx \int_0^\infty |\mathcal{H} k_{f(\sqrt{\mathcal{L}_n})}(t)|^2 |\mathbf{c}(t)|^{-2}\, dt \\
&\approx \int_0^\infty |f(t)|^2 |\mathbf{c}(t)|^{-2}\, dt \lesssim \int_0^\infty |f(t)|^2 (t^2 + t^n)\, dt, \qquad (5.25)
\end{aligned}
$$

which implies that $\mathrm{J} \lesssim (1+\epsilon)\int_0^\infty |f(t)|^2(t^2+t^n)\,dt$. Combining the estimate of I and J yields (ii) and (iii).

The proof for (iv) follows from the same argument except that in (5.25) we use (5.7) instead of (5.6). $\square$

The following decomposition of functions with compact support was proved in [15, Lemma 1.3]; see also [34, Lemma 4.6].

Lemma 5.2. *Let $q, Q \in [0,\infty)$. Suppose that $f \in H^s(\mathbb{R})$ and $\operatorname{supp} f \subset [1/2, 2]$. Then there exist even functions $\{f_\ell\}_{\ell=0}^\infty$ and a positive constant C, independent of f and ℓ, such that for all $\tau \in \mathbb{R}^+$,*

(i) $f(\tau\cdot) = \sum_{\ell=0}^\infty f_{\ell,\tau}(\cdot)$ on $\mathbb{R}^+$, where $f_{\ell,\tau}(\cdot) \equiv f_\ell(\tau\cdot)$ and $\operatorname{supp}\widehat{f_{\ell,\tau}} \subset [-2^\ell\tau, 2^\ell\tau]$;

(ii) for all $\ell \ge 0$ and $\tau \in [1,\infty)$,

$$\int_0^\infty |f_{\ell,\tau}(\xi)|^2(\xi^{2q}+\xi^{2Q})\,d\xi \le C\tau^{-2\min\{q,Q\}-1}2^{-2s\ell}\|f\|^2_{H^s(\mathbb{R})}; \tag{5.26}$$

(iii) for all $\ell \ge 0$ and $\tau \in (0,1)$,

$$\int_0^\infty |f_{\ell,\tau}(\xi)|^2(\xi^{2q}+\xi^{2Q})\,d\xi \le C\tau^{-2\max\{q,Q\}-1}2^{-2s\ell}\|f\|^2_{H^s(\mathbb{R})}. \tag{5.27}$$

Proposition 5.1. *Let m satisfy a mixed Mihlin-Hörmander condition of order (s_0, s_∞) with $s_0 > 3/2$ and $s_\infty > \max\{3/2, (n+1)/2\}$. Then $k_{m(\triangle)}$ satisfies (5.21).*

Proof. Let m_j be as in (5.16). By (5.17), we obtain that for all $x \in S$ and $\epsilon > 0$,

$$\begin{aligned}
&\int_{\epsilon<d(x,e)\le 2\epsilon} \left|k_{m(\triangle)}(x)\right|\,d\rho(x)\\
&\quad= \sum_{\{j\in\mathbb{Z}:\,2^{j/2}\epsilon\ge1\}} \int_{\epsilon<d(x,e)\le 2\epsilon} \left|k_{m_j(2^{-j}\triangle)}(x)\right|\,d\rho(x)\\
&\qquad+ \sum_{\{j\in\mathbb{Z}:\,2^{j/2}\epsilon<1\}} \int_{\epsilon<d(x,e)\le 2\epsilon} \left|k_{m_j(2^{-j}\triangle)}(x)\right|\,d\rho(x) \equiv \mathrm{I}+\mathrm{J}.
\end{aligned}$$

Observe that, by (5.14), $K_{m_j(2^{-j}\triangle)}(x,e) = k_{m_j(2^{-j}\triangle)}(x)$ for all $x \in S$. From this and (5.18), it follows that

$$\begin{aligned}
\mathrm{I} &= \sum_{\{j\in\mathbb{Z}:\,2^{j/2}\epsilon\ge1\}} \int_{\epsilon<d(x,e)\le 2\epsilon} \left|K_{m_j(2^{-j}\triangle)}(x,e)\right|\,d\rho(x)\\
&\lesssim \sum_{\{j\in\mathbb{Z}:\,2^{j/2}\epsilon\ge1\}} \frac{1}{(1+2^{j/2}\epsilon)^\sigma} \int_{\epsilon<d(x,e)\le 2\epsilon} |K_{m_j(2^{-j}\triangle)}(x,e)|(1+2^{j/2}d(x,e))^\sigma\,d\rho(x) \lesssim 1.
\end{aligned}$$

To estimate J, for each $j \in \mathbb{Z}$, set $f^{(j)}(t) \equiv m_j(t^2)$ for all $t \in \mathbb{R}^+$. Since $\operatorname{supp} f^{(j)} \subset [1/2, 2]$, we use Lemma 5.1 to decompose each $f^{(j)}$. Hence, there exists a sequence of even functions

$\{f_\ell^{(j)}\}_{\ell=0}^\infty$ such that $f^{(j)}(2^{-j/2}\cdot)=\sum_{\ell=0}^\infty f_{\ell,2^{-j/2}}^{(j)}(\cdot)$ on $\mathbb{R}^+$ and the Fourier transform of each $f_{\ell,2^{-j/2}}^{(j)}$ is supported in $[-2^{\ell-j/2},2^{\ell-j/2}]$. For simplicity, in the sequel, we denote $f_{\ell,2^{-j/2}}^{(j)}$ by $h_{\ell,j}$. Notice that $m_j(2^{-j}\triangle)=f^{(j)}(2^{-j/2}\sqrt{\triangle})$. Summarizing all these facts, we obtain that for all $j\in\mathbb{Z}$,

$$\int_{\epsilon<d(x,e)\le 2\epsilon}\left|k_{m_j(2^{-j}\triangle)}(x)\right|\,d\rho(x)=\sum_{\ell=0}^\infty\int_{\epsilon<d(x,e)\le 2\epsilon}\left|k_{h_{\ell,j}(\sqrt{\triangle})}(x)\right|\,d\rho(x). \tag{5.28}$$

By the support condition of $\widehat{h_{\ell,j}}$, we have $\operatorname{supp} k_{h_{\ell,j}(\sqrt{\triangle})}\subset B(e,2^{\ell-\frac{j}{2}})$. By this and Lemma 5.2(i), the sum in (5.28) reduces to the sum for ℓ satisfying $2^{\ell-j/2}>\epsilon$, therefore,

$$\mathrm{J}=\sum_{\{j\in\mathbb{Z}:\,2^{j/2}\epsilon<1\}}\ \sum_{\{\ell\in\mathbb{N}:\,2^{\ell-j/2}>\epsilon\}}\int_{\epsilon<d(x,e)\le 2\epsilon}\left|k_{h_{\ell,j}(\sqrt{\triangle})}(x)\right|\,d\rho(x).$$

Set

$$\mathrm{W}_1\equiv\{(j,\ell):\,2^{j/2}\epsilon<1,\,2^{\ell-j/2}>\epsilon,\,\ell\ge 0,\,j>0,\,2^{\ell-j/2}\ge 1\},$$
$$\mathrm{W}_2\equiv\{(j,\ell):\,2^{j/2}\epsilon<1,\,2^{\ell-j/2}>\epsilon,\,\ell\ge 0,\,j>0,\,2^{\ell-j/2}<1\},$$
$$\mathrm{W}_3\equiv\{(j,\ell):\,2^{j/2}\epsilon<1,\,2^{\ell-j/2}>\epsilon,\,\ell\ge 0,\,j\le 0,\,2^{\ell-j/2}\ge 1\}$$

and

$$\mathrm{W}_4\equiv\{(j,\ell):\,2^{j/2}\epsilon<1,\,2^{\ell-j/2}>\epsilon,\,\ell\ge 0,\,j\le 0,\,2^{\ell-j/2}<1\}.$$

Observe that $\mathrm{W}_4=\emptyset$. Correspondingly, for all $k=1,2,3$, we set

$$\mathrm{J}_k\equiv\sum_{(j,\ell)\in\mathrm{W}_k}\int_{\epsilon<d(x,e)\le 2\epsilon}\left|k_{h_{\ell,j}(\sqrt{\triangle})}(x)\right|\,d\rho(x).$$

Thus, $\mathrm{J}=\sum_{k=1}^3\mathrm{J}_k$.

If $(j,\ell)\in\mathrm{W}_1$, then $j>0$ and hence $\epsilon<2^{-j/2}<1$. By this, (5.22), (5.27) and the assumption $s_\infty>\max\{3/2,(n+1)/2\}$, we obtain

$$\begin{aligned}\mathrm{J}_1&\lesssim\sum_{j>0}\sum_{\{\ell\in\mathbb{N}:\,2^{\ell-j/2}\ge 1\}}\epsilon^{(n+1)/2}\left\{\int_0^\infty|h_{\ell,j}(t)|^2(t^2+t^n)\,dt\right\}^{1/2}\\&\lesssim\sum_{j>0}\sum_{\ell\ge j/2}2^{\frac{j}{2}\max\{3/2,(n+1)/2\}}2^{-s_\infty\ell}\|f^{(j)}\|_{H^{s_\infty}(\mathbb{R})}\\&\lesssim\sum_{j>0}\sum_{\ell\ge j/2}2^{\frac{j}{2}(\max\{3/2,(n+1)/2\}-s_\infty)}2^{-s_\infty(\ell-j/2)}\|m\|_{s_\infty}\lesssim\|m\|_{s_\infty}.\end{aligned}$$

If $(\ell,j)\in\mathrm{W}_2$, then $\epsilon\in(0,1)$. Applying (5.22) and (5.27) yields that when $n\ge 2$,

$$\mathrm{J}_2\lesssim\sum_{j=1}^{-2\log_2\epsilon}\sum_{\ell<j/2}\epsilon^{(n+1)/2}\left\{\int_0^\infty|h_{\ell,j}(t)|^2(t^2+t^n)\,dt\right\}^{1/2}$$

$$\lesssim \sum_{j=1}^{-2\log_2 \epsilon} \sum_{\ell=0}^{\infty} \epsilon^{(n+1)/2} 2^{\frac{j}{2}\max\{3/2,(n+1)/2\}} 2^{-s_\infty \ell} \|m\|_{s_\infty} \lesssim \|m\|_{s_\infty};$$

when $n = 1$, by Lemma 5.1(iv), we replace $\{\int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n)\,dt\}^{1/2}$ in the above estimate by $\{\int_0^\infty |h_{\ell,j}(t)|^2 t\,dt\}^{1/2}$, and then a similar argument also implies that $\mathrm{J}_2 \lesssim \|m\|_{s_\infty}$.

Now we assume $n \geq 2$ and estimate J_3. In this case, when $\epsilon \in [1, \infty)$, we use (5.23), (5.26) and $2^{j/2}\epsilon < 1$ to obtain

$$\begin{aligned} \mathrm{J}_3 &\lesssim \sum_{\{j\leq 0:\, 2^{j/2}\epsilon<1\}} \sum_{\ell=0}^{\infty} \epsilon^{3/2} \left\{ \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n)\,dt \right\}^{1/2} \\ &\lesssim \sum_{\{j\leq 0:\, 2^{j/2}\epsilon<1\}} \sum_{\ell=0}^{\infty} \epsilon^{3/2} 2^{3j/4} 2^{-s_0\ell} \|m\|_{s_0} \lesssim \|m\|_{s_0}. \end{aligned} \tag{5.29}$$

When $\epsilon \in (0, 1)$, applying (5.23) and (5.26) yields that

$$\begin{aligned} \mathrm{J}_3 &\lesssim \sum_{j=-\infty}^{0} \sum_{\ell=0}^{\infty} \epsilon^{(n+1)/2} \left\{ \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n)\,dt \right\}^{1/2} \\ &\lesssim \sum_{j=-\infty}^{0} \sum_{\ell=0}^{\infty} 2^{3j/4} 2^{-s_0\ell} \|m\|_{s_0} \lesssim \|m\|_{s_0}. \end{aligned} \tag{5.30}$$

If $n = 1$, Lemma 5.1(iv) implies that we can replace $\{\int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n)\,dt\}^{1/2}$ in (5.29) and (5.30) with $\{\int_0^\infty |h_{\ell,j}(t)|^2 t^2\,dt\}^{1/2}$, and a similar argument also yields $\mathrm{J}_3 \lesssim \|m\|_{s_0}$.

Combining the estimates of J_1 through J_3 yields $\mathrm{J} \lesssim 1$. From this and the estimate of I, we deduce that $k_{m(\triangle)}$ satisfies (5.21), which completes the proof. □

Proof of Theorem 5.1. By Proposition 5.1, the L^2-boundedness of $m(\Delta)$ and the fact that the kernel of $m(\Delta)$ satisfies Hörmander's condition (4.2), the operator $m(\Delta)$ satisfies all the assumptions of Theorem 4.2. Then $(m(\triangle))^*$ satisfies the desired properties. □

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